Fragmentations, Growth-Fragmentations, and Random Structures

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Abstract

In this thesis we study the stochastic model of fragmentation phenomena. We focus on two themes: applications to random laminations of the disk, and growth-fragmentation processes.

In the first part we use fragmentation theory as the principal tool to study Aldous' Brownian triangulation of the disk, that is a random set of non-crossing chords that divide the disk into triangles. We investigate the number of large triangles and the law of the length of the longest chord, and generalize these results to stable laminations. As part of the proof apparatus, we obtain new results on the number of large splitting events of self-similar fragmentations.

The second part concerns growth-fragmentation processes, which describe particle systems in which each particle grows and splits randomly and independently of the others. We prove that the law of a self-similar growth-fragmentation is determined by a cumulant function and its index of self-similarity. We also introduce a new class of growth-fragmentations that are related to Lévy driven Ornstein-Uhlenbeck type processes and prove a law of large numbers for these growth-fragmentations.

Zusammenfassung

In dieser Arbeit diskutieren wir das stochastische Modell von Fragmentierungsphänomenen. Wir konzentrieren uns auf zwei Themen: Anwendungen auf zufällige Laminierungen des Kreises sowie Wachstumsfragmentierungsprozesse.

Im ersten Teil verwenden wir Fragmentierungstheorie als Hauptinstrument, um die Brownsche Triangulation des Kreises zu untersuchen. Letztere ist eine zufällige Menge von sich nicht überschneidenden Sehnen, die den Kreis in Dreiecke unterteilen. Wir untersuchen die Anzahl der grossen Dreiecke sowie die Verteilung der Länge der längsten Sehne und verallgemeinern diese Resultate auf stabilen Laminierungen. Im Verlauf des Beweises erhalten wir neue Ergebnisse über die Anzahl grosser Spaltungen von selbstähnlichen Fragmentierungen.

Wachstumsfragmentierungsprozesse beschreiben Teilchensysteme, in denen sich jedes Teilchen, unabhängig von den anderen Teilchen, mit einer Rate spaltet, die nur von seiner Grösse abhängt. Im zweiten Teil beweisen wir, dass die Verteilung einer selbstähnlichen Wachstumsfragmentierung durch die Kummulantenfunktion und den Index der Selbstähnlichkeit bestimmt wird. Wir führen dann eine neue Klasse von Wachstumsfragmentierungen ein, die in Zusammenhang stehen zu Ornstein-Uhlenbeck ähnlichen Prozessen, die von Lévy Prozessen angetrieben werden. Wir beweisen dann ein Gesetz der grossen Zahlen für diese Wachstumsfragmentierungen.

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To my parents.

Contents

Chapter 1

Introduction

This thesis concerns particle systems in which each particle splits randomly as time passes. Such fragmentation phenomena could be observed widely in nature: biology and population genetics, aerosols, droplets, mining industry, etc. The first studies of fragmentation from a probabilistic point of view are due to Kolmogorov [53] and Filippov [45]. However, the theory of stochastic fragmentation processes is much more recent and the general framework was built mainly by Bertoin [14, 15]. See [17] for a comprehensive monograph. Fragmentations are relevant to other areas of probability theory, such as branching processes, coalescent processes, multiplicative cascades and random trees.

After the introduction, the main body of this thesis is divided into two parts. In the first part, namely Chapter 2, we present an application of fragmentation theory to random non-crossing configurations of the disk. The second part is concerned with growth-fragmentation processes, in which the mass of a fragment may also grow or decay continuously. In Chapter 3, we characterize the laws of self-similar growth-fragmentations. In Chapter 4, we introduce a new type of growth-fragmentations, which are connected with Ornstein-Uhlenbeck type processes, and establish a law of large numbers.

Chapter 2-4 are autonomous and are based respectively on [75], [76] and [77], with a few modifications.

In the rest of the introduction, we give some background and present our major results. The original contributions of the author will be displayed in a frame with a colored background. Throughout this work, we write $\mathbb{N} = \{1, 2, 3, \dots\}$ for the set of positive integers.

1.1 Random laminations and fragmentations

In this section we present the results of Chapter 2. Let us first define the subject of this section.

Definition 1.1.1. *A* **(geodesic) lamination** L *of the (unit) disk* $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$ *is a closed subset* $L \subset \mathbb{D}$*, which can be written as the union of a collection of* **non-crossing** *Euclidean* *chords (that may intersect only at their endpoints). The connected components of* $\mathbb{D}\backslash L$ *are called* **faces** *of* L*. In particular, if all the faces are triangles, then* L *is a* **triangulation***. A* **random (geodesic) lamination** *is a random variable with values in the space of geodesic laminations.*

Laminations can be viewed as the limits of non-crossing configurations of polygons. They possess interesting geometrical properties and are related to other random combinatorial structures such as trees and maps. See [33] and references therein for an overview.

In this section, we shall focus on an important family of random laminations: the **(random) stable laminations** introduced by Kortchemski [54], a special case of which is Aldous' **Brownian triangulation** [3, 2]. We present a study of their geometrical properties, by using their connections with self-similar fragmentations (see Section 1.1.5). As part of the proof apparatus, we develop a new estimation of the number of splits in a fragmentation process (Section 1.1.4), which is of independent interest.

Remark 1.1.2. *It is worth noting that, apart from the stable laminations, there are other interesting random laminations or variations. We mention the recursive triangulation [36], which is also closely related to fragmentations, and random hyperbolic laminations [37], which are composed by hyperbolic chords of the Poincaré disk.*

1.1.1 The Brownian triangulation

To give a precise definition of the Brownian triangulation, let us recall the encoding of a geodesic lamination by a continuous function. Let $g : [0,1] \to [0,\infty)$ be a continuous function with $g(0) = g(1) = 0$. We first define an equivalence relation $\overset{g}{\sim}$ on $[0,1]$:

$$s \overset{g}{\sim} t \quad \text{if and only if} \quad g(s) = g(t) = \min_{r\in[s\wedge t, s\vee t]} g(r).$$

We next define $s \overset{g}{\approx} t$, if $s \overset{g}{\sim} t$ and at least one of the two following properties is satisfied:

1. $\forall r \in (s \wedge t, s \vee t)$, $g(r) > g(s) = g(t)$,
2. $cl_g(s) := \{r \in (0,1) | r \overset{g}{\sim} s\} \subset [s \wedge t, s \vee t]$.

Write $[\mathrm{e}^{i2\pi s}; \mathrm{e}^{i2\pi t}]$ for the chord connecting the two points $\mathrm{e}^{i2\pi s}, \mathrm{e}^{i2\pi t} \in \partial\mathbb{D}$ and set

$$L_g := \bigcup_{s\overset{g}{\approx}t} [\mathrm{e}^{i2\pi s}; \mathrm{e}^{i2\pi t}].$$

Proposition 1.1.3 ([36]). L_g *is a geodesic lamination.*

Let $\mathbf{e} = (\mathbf{e}_s, 0 \le s \le 1)$ be the normalized Brownian excursion (which is, informally speaking, the Brownian motion conditioned to be positive and to take the value 0 at time 1, see Section

12.2 in [71] for a precise definition). Since almost surely the local minima of $\mathbf{e}$ are all distinct, there is the identity

$$\mathcal{L}_{\mathbf{e}} := \bigcup_{s \overset{\mathbf{e}}{\approx} t} [\mathrm{e}^{i2\pi s}; \mathrm{e}^{i2\pi t}] = \bigcup_{s \overset{\mathbf{e}}{\sim} t} [\mathrm{e}^{i2\pi s}; \mathrm{e}^{i2\pi t}],$$

and the random lamination $\mathcal{L}_{\mathbf{e}}$ is a triangulation, called *the Brownian triangulation*, which is a fractal object with Hausdorff dimension 3/2. See [3] and [61]. Figure 1.1 shows a sample of $\mathcal{L}_{\mathbf{e}}$.

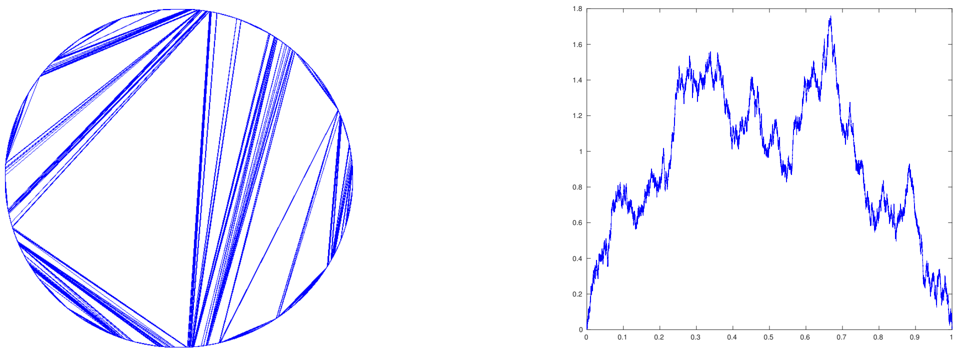

Figure 1.1: **A sample of the Brownian triangulation**

This sample of the Brownian triangulation (left) is encoded by the sample of the Brownian excursion (right).

The universality of the Brownian triangulation For $n \in \mathbb{N}$, let P_n be the polygon formed by the n roots of unity, inscribed in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. A *triangulation* of P_n is the union of its sides and $n-3$ non-crossing diagonals, thus dividing P_n into triangles. A *uniform triangulation* $\mathcal{T}_n$ *of* P_n is a triangulation chosen uniformly at random from the set of all the different triangulations of P_n.

Recall that the Hausdorff metric between two closed subsets $A, B \subset \mathbb{D}$ is

$$d_{Haus}(A, B) := \inf \left\{ \epsilon > 0, A \subset B^{(\epsilon)} \text{ and } B \subset A^{(\epsilon)} \right\},$$

where $A^{(\epsilon)} := \{x \in \mathbb{D} \ : \ d(x, A) < \epsilon\}$ and $B^{(\epsilon)} := \{x \in \mathbb{D} \ : \ d(x, B) < \epsilon\}$. The space of all closed subsets of $\mathbb{D}$ endowed with the Hausdorff distance is a compact metric space. See Section 4.1 in [44] for details. In the seminal work [3], Aldous viewed $\mathcal{T}_n$ as a random closed subset of $\mathbb{D}$, and obtained the Brownian triangulation as the limit object as n tends to infinity.

Theorem 1.1.4 ([3]). *As $n \to \infty$, $\mathcal{T}_n$ converges in distribution to $\mathcal{L}_{\mathbf{e}}$ for the Hausdorff metric.*

Apart from triangulations, there are other non-crossing configurations of polygons. A *dissection* of P_n is the union of its sides and certain non-crossing diagonals, thus dividing P_n into smaller polygons (so not necessarily triangles). A *non-crossing tree* of P_n is a tree drawn on the plane whose vertices are all vertices of P_n. A *non-crossing partition* of P_n is associated with a non-crossing partition of $[n] := \{1, 2, \ldots, n\}$, that is a collection of pairwise disjoint subsets (called *blocks*) whose union is $[n]$, such that for every quadruple $1 \leq i < j < k < \ell \leq n$, if i and k belong to the same block and j and ℓ belong to the same block as well, then i, j, k and ℓ all belong to the same block. They are displayed in Figure 1.2.

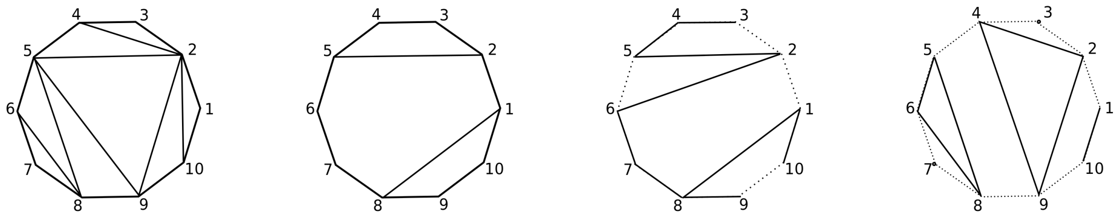

Figure 1.2: **Non-crossing configurations of P_{10}**

From left to right: a triangulation, a dissection, a non-crossing tree and a non-crossing partition (which corresponds to $\{1, 10\}$, $\{2, 4, 9\}$, $\{3\}$, $\{5, 6, 8\}$, $\{7\}$).

The Brownian triangulation is the universal limit of uniformly chosen non-crossing configurations of polygons in the following sense.

Theorem 1.1.5 ([35]). *As n tends to infinity, a uniform dissection, a uniform non-crossing tree or a uniform non-crossing partition converges in distribution to the Brownian triangulation $\mathcal{L}_{\mathbf{e}}$ for the Hausdorff metric.*

Remark 1.1.6. *We mention that if we view a uniform dissection as a random graph by equipping the vertices of the polygons with the graph distance, then it has been proven in [34] that this sequence of random graphs, rescaled by $n^{-\frac{1}{2}}$, convergences in distribution for the Gromov-Hausdorff topology to a multiple of Aldous' Brownian continuum random tree ([1]).*

1.1.2 The number of large faces in the Brownian triangulation

We present now an estimation of the number of "large" faces in the Brownian triangulation $\mathcal{L}_{\mathbf{e}}$, where the loose notion of "large" may have different interpretations. Recall that $\mathcal{L}_{\mathbf{e}}$ is encoded by a normalized Brownian excursion $\mathbf{e}$. Then we write for every $h \geq 0$ that

$$\Theta_{\mathbf{e}}(h) := \{s \in (0,1) : \ \mathbf{e}(s) > h\} = \bigsqcup_{i \geq 1} I_i(h),$$

where $(I_i(h), i \geq 1)$ denote the (disjoint) interval components of the open set $\Theta_{\mathbf{e}}(h)$.

Theorem 1.

1. *For every $\epsilon > 0$, let $N'(\epsilon)$ be the number of triangles in $\mathcal{L}_{\mathbf{e}}$ whose edges have all length greater than ϵ. Then there is*

$$\lim_{\epsilon \to 0} \epsilon N'(\epsilon) = 2 \quad \text{in } L^2(\mathbb{P}).$$

2. Let $N''(\epsilon)$ be the number of triangles in $\mathcal{L}_{\mathbf{e}}$ whose Euclidean area is larger than $\epsilon > 0$. Then there is

$$\lim_{\epsilon \to 0} \epsilon^{\frac{1}{2}} N''(\epsilon) = 4 \int_0^{\infty} \sum_{i=1}^{\infty} \sin(\pi |I_i(s)|) ds \quad \text{in } L^2(\mathbb{P}).$$

These results have been further extended to the stable laminations (see Section 1.1.5). It might be surprising that the limit in case 1 is a constant, while for case 2 is a random variable. We shall see that this phenomenon is in fact an instance of a general phase transition revealed by Theorem 2 in Section 1.1.4. To justify that the limit in case 2 is indeed square integrable, we note that

$$\int_0^{\infty} \sum_{i=1}^{\infty} \sin(\pi |I_i(s)|) ds \leq \pi \int_0^{\infty} \sum_{i=1}^{\infty} |I_i(t)| dt = \pi \int_0^1 \mathbf{e}(s) ds,$$

where the last integral is called the *Brownian excursion area*, which is a random variable with finite k-moment for every $k \in \mathbb{N}$ ([51]).

Our approach to tackle this problem is through fragmentation theory. We can view $(\Theta_{\mathbf{e}}(h), h \geq 0)$ as a fragmentation process, so called **the Brownian fragmentation**, in the sense that as the height h increases, each interval component of $\Theta_{\mathbf{e}}(h)$ splits into smaller intervals. There is a bijection between the faces of $\mathcal{L}_{\mathbf{e}}$ and the splitting events (called **dislocations**) of $\Theta_{\mathbf{e}}$, which is illustrated in Figure 1.3.

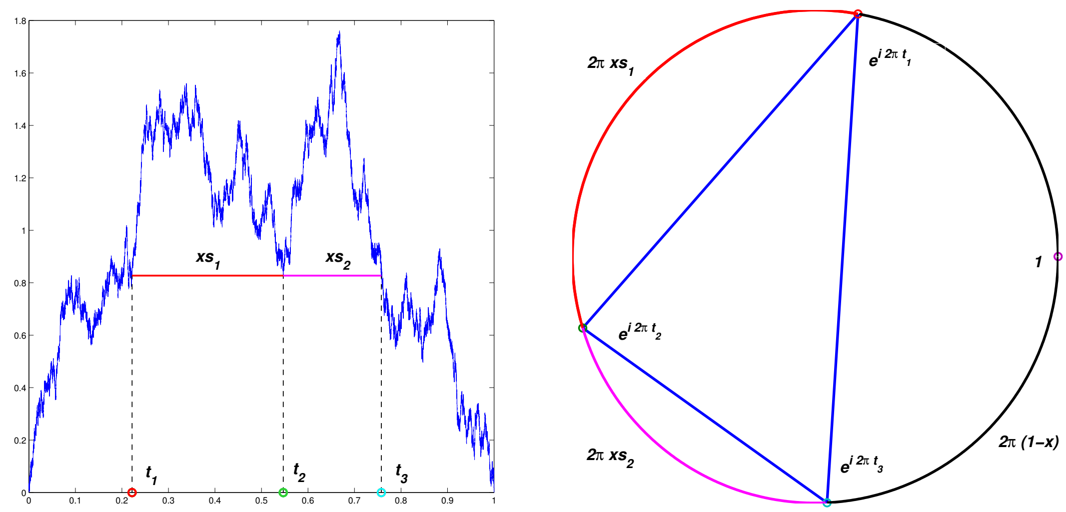

Figure 1.3: **The correspondence between dislocations and triangles.**

The local minimum t_2 of the Brownian excursion $\mathbf{e}$ on the left induces a dislocation of $\Theta_{\mathbf{e}}$, which corresponds to the triangle in $\mathcal{L}_{\mathbf{e}}$ on the right. In this dislocation the interval (t_1, t_3) of length $x = t_3 - t_1$ produces two intervals (t_1, t_2) and (t_2, t_3). Set $s_1 = \max(t_2 - t_1, t_3 - t_2)/x$ and $s_2 = 1 - s_1$, then this dislocation is marked by $(x, (s_1, s_2))$. Since $t_1 \overset{e}{\sim} t_2 \overset{e}{\sim} t_3$, the chords $[\mathrm{e}^{i2\pi t_1}, \mathrm{e}^{i2\pi t_2}]$, $[\mathrm{e}^{i2\pi t_2}, \mathrm{e}^{i2\pi t_3}]$ and $[\mathrm{e}^{i2\pi t_3}, \mathrm{e}^{i2\pi t_1}]$ are included in $\mathcal{L}_{\mathbf{e}}$, and they form a triangle.

This bijection shows that, a dislocation in $\Theta_{\mathbf{e}}$, in which an interval of length $x > 0$ splits into two intervals of lengths (xs_1, xs_2) with $s_1 + s_2 = 1$, corresponds to the triangle in $\mathcal{L}_{\mathbf{e}}$ whose vertices divide the circle into three arcs of lengths $(2\pi(1-x), 2\pi x s_1, 2\pi x s_2)$. Hence with the fragmentation point of view, $N'(\epsilon)$ is the number of those dislocations in Θ_e such that

$$\psi'(x, (s_1, s_2)) := \min(2\sin(\pi x), 2\sin(\pi x s_1), 2\sin(\pi x s_2)) > \epsilon,$$

and $N''(\epsilon)$ is the number of dislocations in Θ_e such that

$$\psi''(x, (s_1, s_2)) := 2\sin(\pi x s_1)\sin(\pi x s_2)\sin(\pi x) > \epsilon.$$

This observation motivates us to develop an estimation of the number of such "large dislocations" for fragmentations, that we shall present in Section 1.1.4, and Theorem 1 will be a consequence. In preparation for that, we need to recall some basic facts about self-similar fragmentations.

1.1.3 Self-similar fragmentation processes

Let us give a short presentation of self-similar fragmentations, of which the materials are gathered from [11, 14, 15, 17]. The reference [17] is a comprehensive monograph of this topic.

Introduce the space of mass-partitions

$$\mathcal{S} := \Big\{\mathbf{s} = (s_1, s_2, s_3, \cdots) : 1 \geq s_1 \geq s_2 \geq \cdots \geq 0,\ \textit{and}\ \sum_{i=1}^{\infty} s_i \leq 1\Big\},$$

which is a compact metric space with ℓ^∞-distance $d(\mathbf{s}, \mathbf{s}') = \sup_{i\in\mathbb{N}} |s_i - s'_i|$.

Definition 1.1.7. *Let $\alpha \in \mathbb{R}$ and $\mathbf{X}^\downarrow = \big(\mathbf{X}^\downarrow(t) := (X_1(t), X_2(t), \ldots),\ t \geq 0\big)$ be a càdlàg Markov process with values in $\mathcal{S}$. For every $x \in [0,1]$, let $\mathbf{P}_x$ denote the law of $\mathbf{X}^\downarrow$ with initial value $\mathbf{X}^\downarrow(0) = (x, 0, \cdots) \in \mathcal{S}$. The process $\mathbf{X}^\downarrow$ is* **a self-similar (mass) fragmentation with index of self-similarity** *α, if it satisfies the following properties:*

1. *(The branching property) For a sequence $\mathbf{x} = (x_1, x_2, \cdots) \in \mathcal{S}$, and every $t \geq 0$, the distribution of $\mathbf{X}^\downarrow$ given $\mathbf{X}^\downarrow(0) = \mathbf{x}$ is the same as the union of the masses, arranged in the decreasing order, of a sequence of independent fragmentations $(\mathbf{X}^{[i]\downarrow})_{i\geq 1}$, where each $\mathbf{X}^{[i]\downarrow}$ has distribution $\mathbf{P}_{x_i}$.*

2. *(The self-similarity) For $x \in [0,1]$, the distribution of the re-scaled process $(x\mathbf{X}^\downarrow(x^\alpha t))_{t\geq 0}$ under $\mathbf{P}_1$ is $\mathbf{P}_x$.*

Without loss of generality, throughout the rest of this section we will implicitly suppose that any fragmentation starts from a single fragment with unit mass, and we will work under $\mathbf{P} := \mathbf{P}_1$.

A fundamental result in fragmentation theory states that, the law of a self-similar fragmentation is characterized by a triple (α, ρ, ν): $\alpha \in \mathbb{R}$ is the index of self-similarity; $\rho \geq 0$ is the *erosion rate*, which describes the speed at which the fragments melt continuously; the σ-finite measure ν on $\mathcal{S}$ is called the *dislocation measure*, which fulfills $\nu(\{(1, 0, \cdots)\}) = 0$ and

$$\int_{\mathcal{S}} (1 - s_1)\nu(d\mathbf{s}) < \infty. \tag{1.1.1}$$

The dislocation measure ν describes the statistics of the fragments induced by a dislocation (a division event). Roughly, for $\mathbf{s} = (s_1, s_2, \cdots) \in \mathcal{S}$, a fragment of mass x splits into masses $(xs_1, xs_2, \cdots)$ at rate $x^\alpha \nu(d\mathbf{s})$. We stress that ν can be an infinite measure, i.e. $\nu(\mathcal{S}) = \infty$, and in that case division events occur instantaneously, in the sense that on every arbitrarily small time interval, infinitely many dislocations occur. But (1.1.1) guarantees that their accumulation does not instantaneously shatter the fragment into dust. The condition (1.1.1) bears the same role as the condition $\int_{(0,\infty)} (1 \wedge x)\Lambda(dx) < \infty$ for a Lévy measure Λ of a subordinator (see [13]). The proof of this fundamental result is based on connections between mass fragmentation processes and processes that take values in the space of partitions of $\mathbb{N}$. This aspect will be not be treated in this manuscript and we refer to [14, 11] and Chapter 2 and 3 in [17] for interested readers.

The Brownian fragmentation in the preceding subsection provides an example of a self-similar fragmentation. Specifically, recall that $\mathbf{e} = (\mathbf{e}_t, 0 \leq t \leq 1)$ denotes the normalized Brownian excursion. For every $h \geq 0$, let $\mathbf{X}^{\downarrow}_{\mathbf{e}}(h)$ be the sequence of lengths, in decreasing order, of the disjoint interval components of the open set $\Theta_{\mathbf{e}}(h) = \{t \in (0,1) : \mathbf{e}(t) > h\}$, then the process $\mathbf{X}^{\downarrow}_{\mathbf{e}}$ is a self-similar fragmentation with characteristics $(-\frac{1}{2}, 0, \nu_{\mathbf{e}})$, where the dislocation measure $\nu_{\mathbf{e}}$ is binary and conservative (so $\nu_{\mathbf{e}}(d\mathbf{s})$-a.e. there is $s_2 = 1 - s_1$ and $s_3 = s_4 = \ldots = 0$) and specified by

$$\nu_{\mathbf{e}}(ds_1) = \frac{2}{\sqrt{2\pi s_1^3 (1 - s_1)^3}} ds_1, \quad 1/2 \leq s_1 < 1.$$

See [15]. The fragmentation $\mathbf{X}^{\downarrow}_{\mathbf{e}}$ is also closely related to the fragmentation introduced by Aldous and Pitman [4] in the study of the standard additive coalescent.

To have a better understanding of the dynamics of self-similar fragmentations, we now present the construction for the basic but important case when a fragmentation process $\mathbf{X}^{\downarrow}$ with characteristics $(\alpha, 0, \nu)$ has finite dislocation measure (i.e. $\nu(\mathcal{S}) < \infty$) and is then specified as *a fragmentation chain*. Without loss of generality, we may assume $\nu(\mathcal{S}) = 1$, and then $\mathbf{X}^{\downarrow}$ is known (Proposition 1.4 in [17]) to possess the following genealogical structure. We index the fragments by the *Ulam-Harris tree* $\mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n$ with $\mathbb{N}^0 = \{\emptyset\}$ by convention, so an element $u \in \mathcal{U}$ is a finite sequence of natural numbers $u = (n_1, \ldots, n_{|u|})$ where $|u| \in \mathbb{N}$ stands for the generation of u. The initial particle denoted by $\emptyset$ is born at time $b_\emptyset = 0$ with mass $a_\emptyset = 1$. After its lifetime $\zeta_\emptyset$, which is an exponential random variable with parameter $(a_\emptyset)^\alpha$, this particle splits into a sequence of particles with masses $(a_i := a_\emptyset \lambda_i, i \in \mathbb{N})$, where $(\lambda_1, \lambda_2, \ldots) \in \mathcal{S}$ has distribution ν. Each child fragment continues in a similar way. See Figure 1.4.

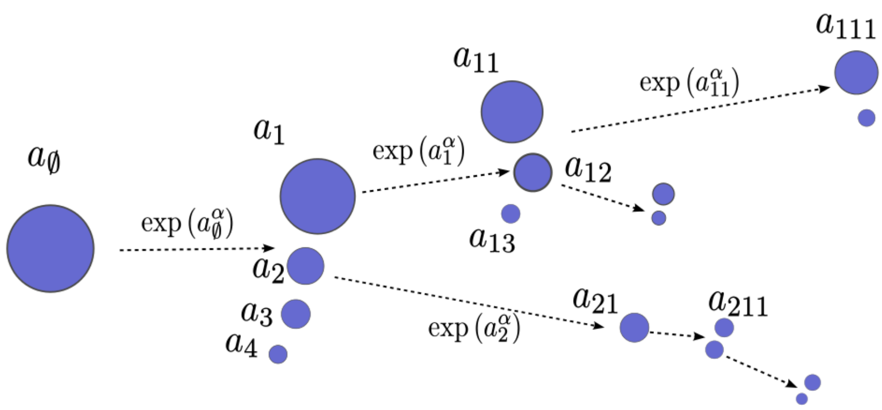

Figure 1.4: **The genealogy of a fragmentation chain**

A formal definition of a fragmentation chain is as follows.

Definition 1.1.8. *Suppose $\nu(\mathcal{S}) < \infty$. Let $(\mathcal{E}_u, u \in \mathcal{U})$ be a family of i.i.d. exponential random variables with parameter $\nu(\mathcal{S})$, $((\lambda_{ui})_{i\in\mathbb{N}}, u \in \mathcal{U})$ be a family of i.i.d. random variables with distribution $\nu(\cdot)/\nu(\mathcal{S})$. The two families are independent. With initial values $b_\emptyset = 0$ and $a_\emptyset = 1$, we define recursively*

$$a_{ui} = a_u \lambda_{ui}, \quad b_{ui} = b_u + \zeta_u, \quad \zeta_{ui} = a_{ui}^{-\alpha} \mathcal{E}_{ui}, \qquad \text{for every } u \in \mathcal{U}, i \in \mathbb{N}.$$

For every $u \in \mathcal{U}$ the triple (a_u, b_u, ζ_u) stand for the mass, the birth time and the lifetime respectively of the particle indexed by u. For every $t \geq 0$, we define $\mathbf{X}^\downarrow(t) = (X_1(t), X_2(t), \ldots)$ by listing the elements of the multiset (which is like a set but allows multiple instances of elements) $\{\!\{a_u : \ u \in \mathcal{U}, \ t \in [b_u, b_u + \zeta_u)\}\!\}$ in decreasing order. Then the process $\mathbf{X}^\downarrow$ is a **fragmentation chain with characteristics** *$(\alpha, 0, \nu)$.*

1.1.4 Large dislocations in a self-similar fragmentation

Let $(\mathbf{X}^\downarrow(t) = (X_1(t), X_2(t), \ldots), t \geq 0)$ be a self-similar fragmentation process with characteristics $(\alpha, 0, \nu)$ (ν is possibly infinite). Motivated by questions about the Brownian triangulation, we consider a bounded function $\varphi : \mathcal{S} \to (0, \infty)$ and a constant $b > 0$, and for every $\epsilon > 0$, we focus on the dislocations in which a fragment of mass $x > 0$ splits into masses $(xs_1, xs_2, \cdots)$ with $\mathbf{s} = (s_1, s_2, \cdots) \in \mathcal{S}$ such that

$$\varphi(\mathbf{s}) x^b > \epsilon.$$

Denote by $N^{(b)}(\epsilon)$ the number of all such dislocations in $\mathbf{X}^{\downarrow}$. We suppose that the dislocation measure ν is *non-lattice* (see Definition 2.2.4 for a precise meaning). We say that $\mathbf{X}^{\downarrow}$ is *conservative* if $\nu\left(\mathbf{s} \in \mathcal{S} : \sum_{i=1}^{\infty} s_i < 1\right) = 0$, and is *dissipative* otherwise, and set

$$m := \begin{cases} \int_{\mathcal{S}} \sum_{i=1}^{\infty} s_i \log(s_i^{-1}) \nu(d\mathbf{s}) & \text{when } \nu \text{ is conservative,} \\ +\infty & \text{when } \nu \text{ is dissipative.} \end{cases}$$

Theorem 2.

In the notation above, suppose that there exist $a \geq 0$ and $C_{\varphi} > 0$ such that

$$g(u) := \nu\left(\mathbf{s} \in \mathcal{S} : \varphi(\mathbf{s}) > u\right) \sim C_{\varphi} u^{-a}, \qquad \text{as } u \to 0^+.$$

Note that if $\nu(\mathcal{S}) < \infty$, then this assumption holds with $a = 0$ and $C_{\varphi} = \nu(\mathcal{S})$.

1. *If $b < \frac{1}{a}$, then*
$$\lim_{\epsilon \to 0} \epsilon^{\frac{1}{b}} N^{(b)}(\epsilon) = \frac{1}{m} \int_0^{\infty} g(u^b) du \quad \text{in } L^2(\mathbb{P}),$$
with convention $\frac{1}{\infty} = 0$.
2. *If $b > \frac{1}{a}$, then*
$$\lim_{\epsilon \to 0} \epsilon^{a} N^{(b)}(\epsilon) = C_{\varphi} \int_0^{+\infty} \sum_{i=1}^{\infty} X_i(t)^{ab+\alpha} dt \quad \text{in } L^2(\mathbb{P}).$$

Theorem 2 is a sightly weaker version of Theorem 2.2.6 in Chapter 2. We see that there is a critical value $\frac{1}{a}$ that depends on ν and φ, such that a phase transition happens. For $b < \frac{1}{a}$, the limit is a finite constant[1]. For $b > \frac{1}{a}$, the limit is a random functional of the fragmentation $\mathbf{X}^{\downarrow}$ and has the same distribution as the *fragmentation area* [18] of a self-similar fragmentation with characteristics $(1 - ab, 0, \nu)$ (see Lemma 2.2.1), which has finite second moment as $1 - ab < 0$. The exact behavior at criticality $b = \frac{1}{a}$ would be an interesting open question.

Theorem 1 is essentially a consequence of Theorem 2. However, Theorem 2 is of independent interest and has further applications. We shall describe some for stable laminations in Section 1.1.5 below.

Notice that if ν is dissipative, then $m = +\infty$ and the limit in case 1 of Theorem 2 is 0. For this situation we obtain a finer result, Theorem 2.5.1, which also has interesting applications. See Section 2.5 for details.

[1] it is indeed finite, since we see from the assumptions that $g(u^b) \sim C_{\varphi} u^{-ab}$ as $u \to 0^+$ and that $g(u) = 0$ for all $u > \|\varphi\|_{\infty}$.

1.1.5 The stable laminations

Kortchemski [54] generalized the Brownian triangulation to a family of random laminations, $(\mathcal{L}_\beta, \beta \in (1,2])$, where each $\mathcal{L}_\beta$ is called the *β-stable lamination*. Similarly to the Brownian triangulation, a stable lamination is also the limit of random dissections of polygons (see Figure 1.2), which are distributed according to a Boltzmann type probability. Specifically, we consider a probability $(\mu_j)_{j\geq 0}$ on $\mathbb{N}\cup\{0\}$ with $\mu_1 = 0$ and mean value 1. Write D_n for the set of all dissections of n-gon P_n. We assign to each dissection $\omega \in D_n$ a weight

$$\pi(\omega) = \prod_{f \text{ face of } \omega} \mu_{\deg(f)-1},$$

and define a probability measure on D_n by normalizing these weights, that is

$$\mathbb{P}_n(\omega) = Z_n^{-1}\pi(\omega),$$

where $Z_n := \sum_{\omega\in D_n} \pi(\omega)$.

Theorem 1.1.9 ([54]). *Let μ be a probability belonging to the domain of attraction of a stable law with exponent $\beta \in (1,2]$ (see e.g. Section 3.7 in [42]) and $\mathcal{D}_n$ be a Boltzmann dissection associated with $\mathbb{P}_n$. Then as n tends to infinity, $\mathcal{D}_n$ converges in distribution for the Hausdorff metric, to a universal limit, which is a random lamination denoted by $\mathcal{L}_\beta$.*

The universal limit $\mathcal{L}_\beta$ is called the **β-stable lamination**. For $\beta = 2$, the 2-*stable lamination* coincides with the Brownian triangulation, i.e. $\mathcal{L}_2 := \mathcal{L}_{\mathbf{e}}$. For $\beta \in (1,2)$, almost surely every face in $\mathcal{L}_\beta$ has infinitely many sides and $\mathcal{L}_\beta$ has Hausdorff dimension $2 - 1/\beta$. See Figure 1.5 for simulations.

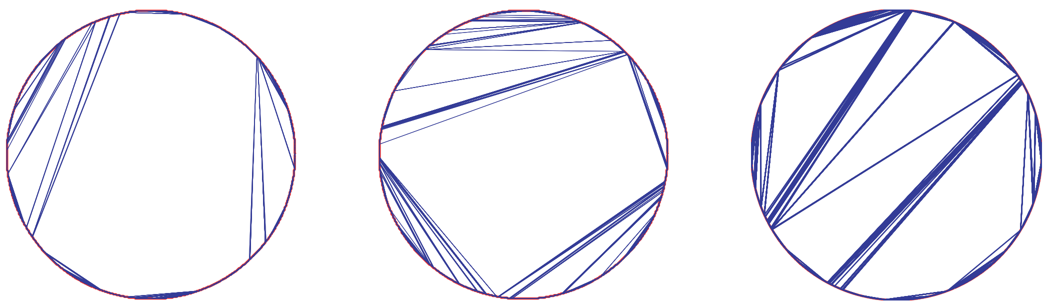

Figure 1.5: **Simulations of stable laminations (by Igor Kortchemski)**
From left to right: $\beta = 1.1$, $\beta = 1.5$ and $\beta = 1.9$.

Let $(H_t^{exc})_{t\in[0,1]}$ be the normalized excursion of the height function of β-strictly stable Lévy process (see Section 2.4 for the precise definition), which is a continuous function with $H_0^{exc} =$

$H_1^{exc} = 0$, then we can encode the β-stable lamination by H^{exc} in the sense of Proposition 1.1.3:

$$\mathcal{L}_\beta \text{ has the same law as } \mathcal{L}_{H^{exc}} := \bigcup_{s \overset{H^{exc}}{\approx} t} [\mathrm{e}^{i2\pi s}; \mathrm{e}^{i2\pi t}].$$

The stable lamination $\mathcal{L}_\beta$ can also be encoded (in a different way) by the normalized excursion of β-strictly stable Lévy process when $\beta \in (1,2)$. See [54] or Section 2.4 for details.

Remark 1.1.10. *Let us give a few remarks on the counterpart of Theorem 1.1.5 for stable laminations. If we sample a random non-crossing partition of $[n]$ (and view it as a lamination) by the Boltzmann weights in a similar way, then as $n \to \infty$, they also converge in distribution to the corresponding β-stable lamination for the Hausdorff metric ([56]). However, for a Boltzmann type random non-crossing tree of P_n, as $n \to \infty$, the limit is not the corresponding β-stable lamination. The limit random lamination is a triangulation with Hausdorff dimension $1 + \frac{1}{\beta}$, and it can be obtained from $\mathcal{L}_\beta$, informally speaking, by "triangulating" each face of $\mathcal{L}_\beta$ from a "uniform" random vertex, i.e. by joining this vertex to each other vertex of the face by a chord. See [55] for details.*

Connections with fragmentations For $\beta \in (1,2)$, we find a bijection between the faces (which are not triangles) in the β-stable lamination and the dislocations in the so-called *β-stable fragmentation* $\mathbf{X}_\beta^\downarrow$, a self-similar fragmentation with index $(1/\beta - 1) < 0$, no erosion and dislocation measure

$$\nu_\beta(d\mathbf{s}) = D_\beta \mathbb{E}\left[T_1; \frac{\Delta T_{[0,1]}}{T_1} \in d\mathbf{s}\right], \tag{1.1.2}$$

where $D_\beta = \frac{\beta^2\Gamma(2-1/\beta)}{\Gamma(2-\beta)}$, $(T_x)_{x\geq 0}$ is a β^{-1}-stable subordinator (see e.g. [13]), which is characterized by its Laplace exponent

$$\mathbb{E}\left[\exp(-qT_x)\right] = \exp\left(-x(\beta\Gamma(1-\beta^{-1}))^{-1}\int_0^\infty r^{-1-1/\beta}(1-\mathrm{e}^{-qr})dr\right), \quad q \geq 0,$$

and $\Delta T_{[0,1]} = (\Delta_1, \Delta_2, \cdots)$ is the vector of jumps of T before time 1 reordered in the decreasing order. The stable fragmentations $(\mathbf{X}_\beta^\downarrow, \beta \in (1,2])$ are studied in [64, 65], see also [47] for a deep study of its behavior near extinction.

Proposition 1.

For $\beta \in (1,2]$, there is a bijection between the dislocations of the fragmentation $\mathbf{X}_\beta^\downarrow$ and the faces in the β-stable lamination. If a dislocation is labeled by $(x, \mathbf{s}) \in [0,1] \times \mathcal{S}$, then this dislocation corresponds to a face whose vertices divide the circle $\partial\mathbb{D}$ into arcs of lengths $2\pi(1-x, xs_1, xs_2, \cdots)$, and the edges of this face have lengths $(2\sin(\pi x), 2\sin(\pi x s_1), 2\sin(\pi x s_2), \cdots)$.

This bijection is intuitively due to the fact that $\mathbf{X}^{\downarrow}_{\beta}$ and $\mathcal{L}_{\beta}$ are both encoded by the same continuous function H^{exc} (see Section 2.4).

The number of large faces We would like to find a result of type Theorem 1. However, almost surely every face in the β-stable lamination has infinitely many edges, hence the shortest edge is always 0. We define alternatively the large faces as follows. For each face, its vertices divide the circle $\partial\mathbb{D}$ into infinitely many arcs, which shall be referred to as *the arcs of the face.*

Theorem 3.

For $\beta \in (1,2)$, let $N(\epsilon)$ be the number of those faces in the β-stable lamination, such that at least two of the arcs of the face are longer than ϵ, and the total length of the remaining arcs is greater than ϵ. Then

$$\lim_{\epsilon\to 0} \epsilon N(\epsilon) = \frac{2\pi(\beta-1)}{\Gamma(2-\beta)}\mathbb{E}\left[\min(T_1-\Delta_1,\Delta_1)\right] \quad \text{in } L^2(\mathbb{P}), \tag{1.1.3}$$

where T_1 is the value of the β^{-1}-stable subordinator T at time 1, and Δ_1 is the largest jump of T before time 1.

Recall that the dislocation measure ν_β as in (1.1.2) fulfills (1.1.1), we see that the limit in (1.1.3) is indeed finite:

$$\mathbb{E}\left[\min(T_1-\Delta_1,\Delta_1)\right] \leq \mathbb{E}\left[T_1-\Delta_1\right] = D_\beta^{-1}\int_{\mathcal{S}}(1-s_1)\nu_\beta(d\mathbf{s}) < \infty. \tag{1.1.4}$$

It would be an interesting open question to calculate explicitly the value of the limit. See a discussion of the joint distribution of (T_1,Δ_1) in Section 2.4 (after Theorem 2.4.2).

The length of the longest chord Write $2\pi A_\beta$ for the length of the minor arc with the same endpoints as the longest chord in $\mathcal{L}_\beta$, so $A_\beta \leq \frac{1}{2}$ and the longest chord has length $2\sin(\pi A_\beta)$. For the Brownian triangulation, Aldous has found the law of A_2 by calculating the length of the longest chord of the uniform triangulation $\mathcal{T}_n$ of a polygon P_n and using the convergence Theorem 1.1.5. His result (formula (9) in [3]) shows that the distribution function of the random variable A_2 is

$$\mathbb{P}\left(A_2<a\right) = 6\pi^{-1}(\arctan(3^{-\frac{1}{2}}) - \arctan((1-2a)^{\frac{1}{2}})) - \frac{(3a-1)(1-2a)^{\frac{1}{2}}}{\pi a(1-a)}, \quad \frac{1}{3}<a<\frac{1}{2}.$$

We obtain new results for the stable laminations.

Proposition 2.

For $\beta \in (1,2)$, in the β-stable lamination, A_β has distribution function

$$\mathbb{P}(A_\beta \leq a) = \frac{\beta\Gamma(2-\beta^{-1})}{\Gamma(2-\beta)\Gamma(1-\beta^{-1})}\mathbb{E}\left[T_1\mathbf{1}_{\left\{\frac{\Delta_1}{T_1}<\frac{a}{1-a}\right\}}\int_{\frac{\Delta_1}{aT_1}\vee 1}^{\frac{1}{1-a}}(1-x^{-1})^{-1/\beta}dx\right], \quad 0<a<\frac{1}{2}.$$

If a Boltzmann dissection $\mathcal{D}_n$ of P_n converges as $n \to \infty$ to $\mathcal{L}_\beta$ in the sense of Theorem 1.1.9, then the length of the longest diagonal of $\mathcal{D}_n$ converges in distribution towards the law of $2\sin(\pi A_\beta)$. Note that if $\frac{\Delta_1}{T_1} < \frac{a}{1-a}$, then $T_1 < \frac{1-a}{1-2a}(T_1 - \Delta_1)$ and hence it follows from (1.1.4) that the expected value in the formula above in indeed finite.

Our approach relies on the bijection in Proposition 1. Specifically, noticing that the longest chord is an edge of the *centroid*, the (almost surely) unique face that contains the origin, we will answer this question by exploring the dislocation associated with the centroid. It is not difficult to see that the dislocation in Θ_β associated with the centroid is the unique dislocation, in which $x > 0$ splits into a sequence $(x\mathbf{s})$ with $\mathbf{s} \in \mathcal{S}$, such that $\min(x, 1-xs_1) > \frac{1}{2}$. As a consequence, for every $a \in (0, \frac{1}{2}]$ we have that

$$\mathbb{P}(A_\beta < a) = \mathbb{E}[N(1-a)],$$

where $N(1-a)$ is the number of dislocations in Θ_β such that $\min(x, 1-xs_1) > 1-a$. Using this observation, we hence recover Aldous' formula for the Brownian triangulation and obtain Proposition 2 for the stable laminations.

1.2 Self-similar growth-fragmentation processes

We present in this section the results developed in Chapter 3.

We consider *growth-fragmentation processes* [19, 20], which describe the random evolution of particles which not only split, but also grow or decay continuously as time passes, independently one of the others. Similar to Definition 1.1.7, **a self-similar growth-fragmentation with index of self-similarity** $\alpha \in \mathbb{R}$ is a Markov process

$$\mathbf{X}^{(\alpha)\downarrow} = \left(\mathbf{X}^{(\alpha)\downarrow}(t) := (X_1(t), X_2(t), \ldots),\ t \geq 0\right)$$

that possesses a càdlàg path in $c_0^\downarrow$, the space of decreasing null-sequences endowed with the ℓ^∞-norm. As the total sum of masses of particles may vary in a growth-fragmentation, here the space of mass-partitions $\mathcal{S}$ is not suitable as state space, so it is replaced by the space $c_0^\downarrow$.

Further, $\mathbf{X}^{(\alpha)\downarrow}$ satisfies the following properties. For every $x \in [0, \infty)$, let $\mathbf{P}_x$ denote the law of $\mathbf{X}^{(\alpha)\downarrow}$ with initial value $\mathbf{X}^{(\alpha)\downarrow}(0) = (x, 0, \cdots) \in c_o^\downarrow$.

(P1) (The branching property) For a sequence $\mathbf{x} = (x_1, x_2, \cdots) \in c_o^\downarrow$, and every $t \geq 0$, the distribution of $\mathbf{X}^{(\alpha)\downarrow}$ with $\mathbf{X}^{(\alpha)\downarrow}(0) = \mathbf{x}$ is the same as the union of the masses, arranged in the decreasing order, of a sequence of independent fragmentations $(\mathbf{X}^{[i]\downarrow})_{i\geq 1}$, where each $\mathbf{X}^{[i]\downarrow}$ has distribution $\mathbf{P}_{x_i}$.

(P2) (The self-similarity) There exists $\alpha \in \mathbb{R}$, such that for $x \in [0, \infty)$, the distribution of the re-scaled process $(x\mathbf{X}^{(\alpha)\downarrow}(x^\alpha t))_{t\geq 0}$ under $\mathbf{P}_1$ is $\mathbf{P}_x$.

When $\alpha = 0$, $\mathbf{X}^{(0)\downarrow}$ is called a *homogeneous growth-fragmentation.*

Self-similar growth-fragmentations are connected with certain family of random planar maps, see [22, 21].

Bertoin [20] developed a general construction of growth-fragmentations, so-called *Markovian growth-fragmentation processes*, which in particular can be used to build self-similar growth-fragmentations. The major task in this section is to characterize the laws of self-similar growth-fragmentations built in this way.

1.2.1 Markovian growth-fragmentation processes

Let us first recall the approach in [20]. It is convenient to describe a Markovian growth-fragmentation as a cell system, in which each cell may grow continuously and divide into two cells occasionally. These dynamics, both the growth and the splitting, are encoded by a càdlàg Markov process $X = (X(t), t \geq 0)$ on $[0, \infty)$ with no positive jumps, which shall be referred to as a **cell process**. Specifically, at initial time 0 there exists a single cell, called the *Eve*. As time proceeds, the size of Eve evolves according to the cell process X. At each jump time $t \geq 0$ of X with $\Delta X(t) = X(t) - X(t-) < 0$, a "daughter" cell with initial size $-\Delta X(t)$ is born. We stress that the Eve survives after this cell division. Each daughter follows the same dynamics as the Eve and evolves independently of the others.

This description can be made rigorous. For simplicity, we may assume $\lim_{t\to\infty} X(t)$ exists (which is not essential, see Section 3.2.3), then we can list the jump times in decreasing order of the size of the jumps. We next index the cell system by the Ulam-Harris tree $\mathcal{U}$ and build for each $u \in \mathcal{U}$ a process $\mathcal{X}_u$ that depicts the evolution of the size of the cell indexed by u as time passes, in the following way. For $y > 0$, write P_y for the law of X starting from $X(0) = y$.

Definition 1.2.1 ([20]). *For $x > 0$, a* **cell system** *$\mathcal{X} := (\mathcal{X}_u,\ u \in \mathcal{U})$ driven by X, in which the Eve cell $\emptyset$ has the initial size x, is built by the following description.*

1. *(Eve) We set the birth time of $\emptyset$ by $b_\emptyset := 0$ and let the Eve process $\mathcal{X}_\emptyset = (\mathcal{X}_\emptyset(t), t \geq 0)$ be of law P_x.*

2. *For every individual $u \in \mathcal{U}$ ans $i \in \mathbb{N}$, conditionally on $\mathcal{X}_u$, say the i-th largest (in size) jump of $\mathcal{X}_u$ occurs at time t_i and has size $x_i := -\Delta\mathcal{X}_u(t)$, then its i-th daughter ui is born at time $b_{ui} := t_i$ and ui's size process $\mathcal{X}_{ui} = (\mathcal{X}_{ui}(r), r \geq 0)$ has conditional distribution P_{x_i}, independent of the size processes of the other individuals in the same generation.*

Write $\mathcal{P}_x$ for the law of this cell system $\mathcal{X}$ (recall that $x > 0$ indicates the initial size of the Eve $\emptyset$, i.e. $\mathcal{X}_\emptyset(0) = x$). According to [50], the probability distribution $\mathcal{P}_x$ indeed exists and is uniquely determined by the above description.

Definition 1.2.2 ([20]). *Let $\mathcal{X}$ be a cell system driven by X. For every $t \geq 0$, the multiset of the sizes of the cells alive at time t is*

$$\mathbf{X}(t) := \{\!\{\mathcal{X}_u(t - b_u) \ : \ u \in \mathcal{U}, b_u \leq t\}\!\},$$

where b_u is the birth time of u. Then we call $\mathbf{X} := (\mathbf{X}(t), t \geq 0)$ a **(Markovian) growth-fragmentation process associated with the cell process** *X and we write $\mathbf{P}_x$ for the law of $\mathbf{X}$ under $\mathcal{P}_x$.*

To understand the evolution of this model and give a first example, let us go back to a (homogeneous) fragmentation chain $\mathbf{X}^{\downarrow}$ with characteristics $(0, 0, \nu)$ defined as in Definition 1.1.8, such that ν has support on $\{\mathbf{s} \in \mathcal{S} : \ s_1 + s_2 = 1\}$ (binary). However, we consider a different genealogical structure: for each splitting event when a fragment a_u with $u \in \mathcal{U}$ splits into two fragments a_{u1} and a_{u2}, we view this event as a mother particle gives birth to a single child particle with mass a_{u2}. We stress that the mother particle survives after this birth event, and has mass a_{u1} immediately after this birth event. Therefore, we obtain a cell system description. See Figure 1.6 for an illustration. We summarize this observation in the following example.

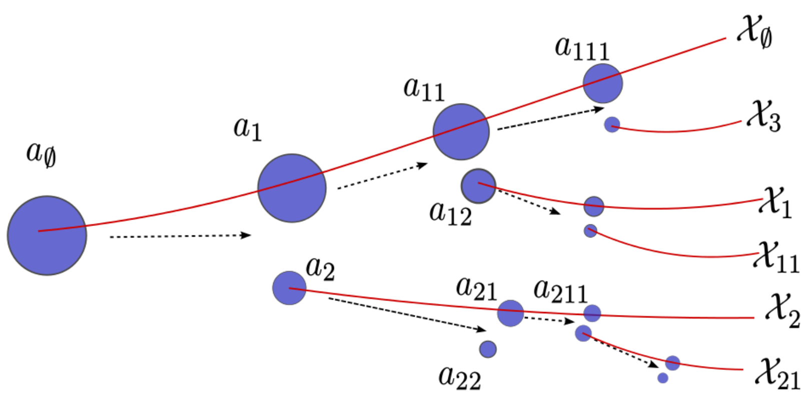

Figure 1.6: **A fragmentation chain viewed as a cell system**

Example 1.2.3. *Let $\mathbf{X}^{\downarrow}$ be a binary homogeneous fragmentation chain with characteristics $(0, 0, \nu)$, such that ν has support on $\{\mathbf{s} \in \mathcal{S} : \ s_1 + s_2 = 1\}$. Then the process X associated with*

$\{1, 1, 1, \ldots\}$ *($\mathcal{X}_\emptyset$ in Figure 1.6) is the exponential of a compound Poisson process with Lévy measure Λ_1 given by the image of ν by the map $\mathbf{s} \mapsto s_1$. Similarly, the process Y associated with* $\{2, 2, 2, \ldots\}$ *is the exponential of a compound Poisson process with Lévy measure Λ_2 given by the image of ν by the map $\mathbf{s} \mapsto s_2$. Then the homogeneous fragmentation $\mathbf{X}^{\downarrow}$ can be viewed as a Markovian growth-fragmentation associated with either X or Y.*

This example also shows that, although by construction, the law of a growth-fragmentation is determined by the law of the cell process, growth-fragmentations driven by cell processes with different laws may have the same distribution. To understand this phenomenon is the main object of this section that will be discussed in detail in Section 1.2.2 and Section 1.2.4.

Unlike the pure fragmentation case, in which the sum of the masses of the fragments in the system can only decay, a growth-fragmentation $\mathbf{X}$ may *explode*, this is to say, at a finite time $t \geq 0$, the multiset $\mathbf{X}(t)$ might be not locally finite. Bertoin [20] proposed a practical non-explosion criterion, a slightly more general version of which is as follows. Let us fix a measurable function $f \colon [0, \infty) \times (0, \infty) \to [0, \infty)$ which fulfills

$$\inf_{r<l, x>a} f(r, x) > 0, \qquad \text{for every } a, l > 0. \tag{1.2.1}$$

We suppose that X satisfies

[H] for every $x > 0$ and every $s, t \geq 0$, there is

$$E_x \left[f(s+t, X(t)) + \sum_{0 \leq r \leq t} f(s+r, -\Delta X(r)) \right] \leq f(s, x).$$

When f only depends on the x variable, i.e. $f(t, x) \equiv f(x)$ for every $x, t \geq 0$. In that case f is a so-called *excessive function* for $\mathbf{X}$, see [20]. Let us define for every $s \geq 0$ a space $\mathcal{M}^s_f$: a multiset $\mathcal{I} \in \mathcal{M}^s_f$, if $\mathcal{I}$ has elements in $(0, \infty)$ and $\langle \mathcal{I}, f(s, \cdot) \rangle < \infty$.

Proposition 3 (Modification of Theorem 1 in [20]).

Suppose that X satisfies **[H]**, *then for every $x > 0$ and $t \geq 0$, the multiset $\mathbf{X}(t) \in \mathcal{M}^t_f$, $\mathbf{P}_x$-almost surely.*

So we can list the elements of $\mathbf{X}(t)$ in decreasing order and obtain a null-sequence $\mathbf{X}^{\downarrow}(t) \in c_0^{\downarrow}$. With no risk of confusion, we also call $\mathbf{X}^{\downarrow}$ a **Markovian growth-fragmentation associated with** X. The following statement is essentially a consequence of Proposition 3.

Proposition 1.2.4 (Proposition 2 in [20]). *Suppose that X satisfies* **[H]**, *then $\mathbf{X}^{\downarrow}$ satisfies the branching property* **(P1)**.

1.2.2 The homogeneous case

In this section we focus on homogeneous growth-fragmentations, that fulfill **(P2)** with $\alpha = 0$. This case is closely related to Lévy processes (a càdlàg process with independent and stationary increments). We refer to [12, 58] for general theory of Lévy processes. Let ξ be a Lévy process with no positive jumps, possibly killed at some independent exponential time. Such a process is often referred to as a *spectrally negative Lévy process (SNLP)*. The distribution of the SNLP ξ is characterized by its Laplace exponent $\Phi : [0, \infty) \to \mathbb{R}$:

$$\mathbb{E}\left[\mathrm{e}^{q\xi(t)}\right] = \mathrm{e}^{\Phi(q)t}, \qquad \text{for all } q, t \geq 0.$$

It is well-known that the convex function Φ is can be expressed by the Lévy-Khintchine formula

$$\Phi(q) = -k + \frac{1}{2}\sigma^2 q^2 + cq + \int_{(-\infty,0)} (\mathrm{e}^{qz} - 1 + q(1 - \mathrm{e}^z))\, \Lambda(dz), \quad q \geq 0, \tag{1.2.2}$$

where $k \geq 0$ is the killing rate, $\sigma \geq 0$, $c \in \mathbb{R}$ and the Lévy measure Λ on $(-\infty, 0)$ satisfies

$$\int_{(-\infty,0)} (z^2 \wedge 1)\Lambda(dz) < \infty. \tag{1.2.3}$$

Then we say ξ is a SNLP with characteristics (σ, c, Λ, k).

For $x > 0$, denote by P_x the law of the homogeneous cell process

$$X^{(0)} = x \exp(\xi).$$

Let us introduce an important function $\kappa \colon [0, \infty) \to (-\infty, \infty]$

$$\kappa(q) := \Phi(q) + \int_{(-\infty,0)} (1 - \mathrm{e}^z)^q \Lambda(dz), \quad q \geq 0. \tag{1.2.4}$$

Note that κ is convex, $\kappa \geq \Phi$, and $\kappa(q) < \infty$ for all $q \geq 2$ because of (1.2.3). We stress that κ does not characterize the law of ξ, see Lemma 3.2.1 in Chapter 3. We call κ the *cumulant* of ξ or $X^{(0)}$.

Fix $q \geq 2$ and $K \geq \kappa(q)$, we have that $X^{(0)}$ satisfies **[H]** with the function $(t, x) \mapsto x^q \mathrm{e}^{-Kt}$, hence it follows from Proposition 3 that the Markovian growth-fragmentation $\mathbf{X}^{\downarrow}$ associated with $X^{(0)}$ satisfies **(P1)** and takes values in $\ell^{2\downarrow}$ (the subspace of $c_0^{\downarrow}$ of square-summable decreasing sequences endowed with ℓ^2-norm). Moreover, one can easily deduce from the cell system description that $\mathbf{X}^{\downarrow}$ is homogeneous, see Lemma 1 in [20]. We further know that it has a càdlàg version in $\ell^{2\downarrow}$ (Proposition 2 in [19]). So we call $\mathbf{X}^{\downarrow}$ a *homogeneous growth-fragmentation.*

Characterization of the law The binary fragmentation chain in Example 1.2.3 is a homogeneous growth-fragmentation. In this example, we saw an instance where two cell processes which have different distributions, generate the same binary fragmentation chain. Pitman and Winkel [70] studied in depth this phenomenon for the case when X is the exponential of the negative of a pure-jump subordinator (called *fragmenter* by them). Extending their ideas, we characterize the laws of homogeneous growth-fragmentations. Let X and $\tilde{X}$ be homogeneous processes with respective cumulants κ and $\tilde{\kappa}$ defined as in (1.2.4), $\mathbf{X}^{\downarrow}$ and $\tilde{\mathbf{X}}^{\downarrow}$ be growth-fragmentations associated respectively with X and $\tilde{X}$.

Theorem 4.

Two homogeneous growth-fragmentations $\mathbf{X}^{\downarrow}$ and $\tilde{\mathbf{X}}^{\downarrow}$ have the same law if and only if $\kappa = \tilde{\kappa}$. Further, for every $t \geq 0$ with $\mathbf{X}^{\downarrow}(t) = (X_1(t), X_2(t), \ldots)$, we have

$$\mathbb{E}\left[\sum_{i=1}^{\infty} X_i(t)^q\right] = \exp\left(\kappa(q)t\right), \qquad \textit{for all } q \geq 2. \tag{1.2.5}$$

This result has been partially proved in [20]. The identity (1.2.5) can be viewed as the counterpart of the following result for a homogeneous (pure) fragmentation $\mathbf{X}^{\downarrow}$ with characteristics $(0, \rho, \nu)$. One can define the *cumulant* $\kappa : [0, \infty) \to \mathbb{R}$ by

$$\kappa(q) = -\rho q + \int_{\mathcal{S}} \Big(\sum_{i=1}^{\infty} s_i^q - 1\Big)\nu(d\mathbf{s}), \quad q \geq 1,$$

and it follows from Theorem 3 in [15] that

$$\mathbb{E}\left[\sum_{i=1}^{\infty} X_i(t)^q\right] = \exp\left(\kappa(q)t\right) \quad \text{for every } t \geq 0.$$

Let us present a heuristic proof of Theorem 4. Consider a cell system $\mathcal{X}$ associated with X (Definition 1.2.1). Conditionally on $\mathcal{X}$, we first introduce at each division event a certain Bernoulli random variable B, independently of everything else, whose parameter is determined by a measurable function $p : (0, 1) \to [0, 1]$ as follows. If the size (at birth) of the child is $y > 0$ and the size of the parent (immediately before the birth event) is $x > 0$, then the parameter of B is $p(y/x)$. We next change the genealogy of $\mathcal{X}$ according to this family of Bernoulli random variables. Specifically, at each birth event, if the corresponding Bernoulli random variable $B = 1$, then we exchange the roles of the parent and the child; if $B = 0$, then we do nothing.

In this way we get a new cell system $\mathcal{X}^{[p]}$ associated with a new cell process $X^{[p]}$, whose law is determined by the function p and the law of X. Further, it is intuitively clear (however non-trivial) that $\mathcal{X}$ and $\mathcal{X}^{[p]}$ correspond to the same growth-fragmentation $\mathbf{X}$ despite of having different genealogies. So $\mathbf{X}$ can be viewed as a Markovian growth-fragmentation driven by either X or $X^{[p]}$. See Figure 1.7.

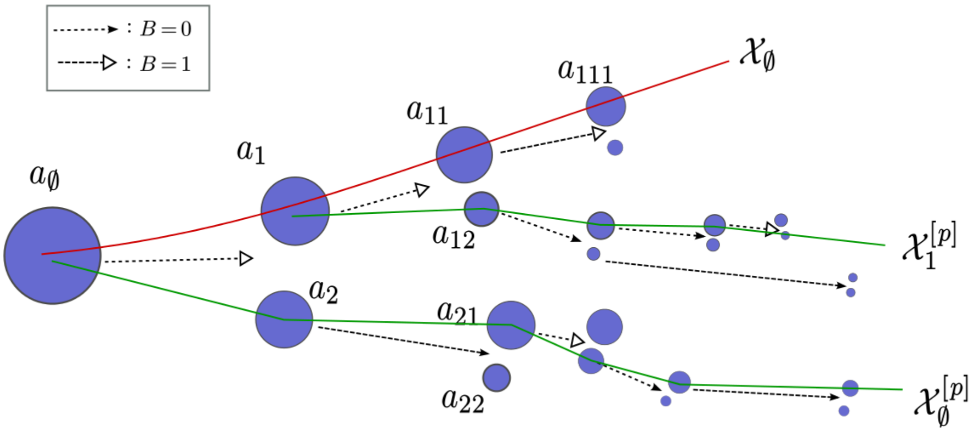

Figure 1.7: **Change of genealogy**

We consider this homogeneous fragmentation chain as a cell system associated with X: in each division, the larger fragment is the parent and the smaller fragment is the child, see also Figure 1.6. Each arrow stands for a division, and the type of the arrow represents the value of its corresponding Bernoulli random variable B. If $B = 1$, then we exchange the roles of the parent and the child; if $B = 0$, then we do nothing. So this fragmentation chain can also be viewed as a cell system $\mathcal{X}^{[p]}$ associated with the new cell process $X^{[p]}$.

The process $X^{[p]}$ is analogous to *Markovian paths* in homogeneous fragmentations (see [70]), and also closely related to *switching transformations* (see [70] and Chapter 3). By the Poissonian structure of Lévy processes, we identify the law of $X^{[p]}$. In particular, we find that $X^{[p]}$ is a homogeneous cell process and has the same cumulant κ (defined as in (1.2.4)) as X. Conversely, if Y is a homogeneous cell process with cumulant κ, then there exists a function p_Y such that the new cell system transformed from $\mathcal{X}$ according to p_Y is driven by Y. We hence conclude that if $\kappa = \tilde{\kappa}$, then $\mathbf{X}^{\downarrow}$ and $\tilde{\mathbf{X}}^{\downarrow}$ have the same law. Combining this with Theorem 1 in [19], which essentially entails (1.2.5), we thus complete the proof of Theorem 4.

The heuristic approach addressed above can be possibly transferred to the self-similar case, as we shall present in the next section. Note that, however, the rigorous proof of Theorem 4 given in Chapter 3 adapts a different approach, by using a discretization technique (truncating the smaller fragments in each dislocation) and a connection with *branching Lévy processes* [19]. Nevertheless, the underlying ideas are essentially similar.

1.2.3 Self-similar growth-fragmentation processes

To construct self-similar growth-fragmentations in general, we first provide some background on self-similar Markov processes and Lamperti's representation [59] of the latter.

Non-negative self-similar Markov processes Let $\alpha \in \mathbb{R}$ and ξ be a SNLP. We define a time-change by

$$\tau_t^{(\alpha)} := \inf\left\{ r \geq 0 \ : \ \int_0^r \exp(-\alpha\xi(s))ds \geq t \right\}, \quad t \geq 0.$$

For every $x > 0$, denote by P_x the law of the process

$$X(t) := x \exp(\xi(\tau_{tx^\alpha}^{(\alpha)})), \quad t \geq 0,$$

then for every $\gamma > 0$,

$$\text{the law of } (\gamma X(\gamma^\alpha t), t \geq 0) \text{ under } P_x \text{ is } P_{\gamma x}.$$

So we call X a *self-similar Markov process with index* α. If the SNLP ξ has characteristics (σ, c, Λ, k) and Laplace exponent Φ as in (1.2.2), then we say that X has characteristics $(\sigma, c, \Lambda, k, \alpha)$, or simply (Φ, α). Recall that the cumulant κ of ξ is defined by (1.2.4), then κ is also called the cumulant of the self-similar Markov process X.

Self-similar growth-fragmentations We next consider a non-negative self-similar Markov process X with index $\alpha \neq 0$ and cumulant κ, and we further assume that

$$\text{there exists } q_0 > 0 \text{ with } \kappa(q_0) < 0. \tag{1.2.6}$$

Under this assumption, X satisfies **[H]** for the function $x \mapsto x^{q_0}$. So it follows from Proposition 3 that a Markovian growth-fragmentation $\mathbf{X}^\downarrow$ associated with X is well-defined and takes values in $\ell^{q_0 \downarrow}$ (the subspace of all elements in $c_0^\downarrow$ with finite ℓ^{q_0}-norm). We call $\mathbf{X}^\downarrow$ a *self-similar growth-fragmentation process* with index α. Indeed, one can deduce respectively from Proposition 1.2.4 and the cell system description that $\mathbf{X}^\downarrow$ satisfies the branching property **(P1)** and the self-similarity **(P2)** (with index α), see Theorem 2 in [20]. The process $\mathbf{X}^\downarrow$ has almost surely càdlàg trajectories in $c_0^\downarrow$ (Corollary 4 in [20]). [2]

Remark 1.2.5. *When* (1.2.6) *is not satisfied, more precisely, if $\alpha \neq 0$ and $\kappa(q) > 0$ for all $q \geq 0$, then the self-similar growth-fragmentation $\mathbf{X}^\downarrow$ with index α explodes in finite time. See a discussion in depth in [26].*

Many-to-one formula For the self-similar case with index $\alpha \neq 0$, there does not exist a cumulant in the sense of (1.2.5). Nevertheless, to describe the mean value of the particles, one can use a *one particle picture* developed in [21], which extends Corollary 2 in [15] for self-similar (pure) fragmentations. Suppose that (1.2.6) holds and that ξ is not the negative of a

[2] From Corollary 4 in [20] and the notation therein we know that it has càdlàg path in $\ell_{\mathcal{U}}^{q_0}$ a.s., which implies the càdlàg path in $c_0^\downarrow$ a.s. Indeed, for any $x, y \in \ell_{\mathcal{U}}^{q_0}$, write $x^\downarrow$ and $y^\downarrow$ respectively for the corresponding sequence in decreasing order, then there is $\|x^\downarrow - y^\downarrow\|_{\ell^\infty} \leq \|x - y\|_{\ell^\infty} \leq \|x - y\|_{\ell^{q_0}}$, where the first inequality follows from Theorem 3.5 in [62].

subordinator. Then there exists ω^+ such that $\kappa(\omega^+) = 0$, $\kappa'(\omega^+) > 0$ and $\kappa(q) < \infty$ in some neighborhood of ω^+. Let

$$\Phi^+(q) := \kappa(q + \omega^+), \quad q \geq 0,$$

then there exists a self-similar Markov process Y^+ with characteristics (Φ^+, α).

Theorem 1.2.6 (Theorem 3.5 in [21]). *Let g be a continuous function on $(0, \infty)$ with compact support. Then for every $x > 0$ and $t \geq 0$, there is the identity*

$$\mathbf{E}_x \left[\sum_{i=1}^{\infty} g(X_i(t)) X_i(t)^{\omega^+} \right] = x^{\omega^+} E_x \left[g(Y^+(t)) \right], \tag{1.2.7}$$

where E_x denotes the mathematical expectation under the law of Y^+ started from $Y^+(0) = x$.

1.2.4 Characterization of the laws of self-similar growth-fragmentations

In order to characterize the laws of self-similar growth-fragmentations, we investigate Markovian growth-fragmentations in general and show that if two cell processes X and Y have the marginal distributions of a *bifurcator* in the following sense, then they generate the same (in the sense of finite-dimensional distributions) growth-fragmentation.

Informal Definition 1.2.7 ([70]). *A pair of two Markov processes (X, Y) is a* **bifurcator**, *if they almost surely coincide for a strictly positive time $\tau > 0$ and evolve independently afterwards, with*

$$X(\tau) + Y(\tau) = X(\tau-) = Y(\tau-).$$

This will be stated rigorously in Definition 3.3.7 in Chapter 3. The idea of bifurcator goes back to Pitman and Winkel [70], who explicitly constructed bifurcators of fragmenters and characterized their laws. In Example 1.2.3, the two Eve processes (X, Y) therein form such a bifurcator.

Intuitively speaking, if X and Y have the marginal distributions of a bifurcator, then we can change the genealogy of a cell system $\mathcal{X}$ driven by X, by exchanging the roles of the mother and the child for certain birth events according to a rule, such that the new cell system $\mathcal{Y}$ obtained in this way is associated with Y. This generalizes the idea of the proof of Theorem 4 (the homogeneous case).

Technically, the crucial property that needs to be verified is that the two cell systems $\mathcal{X}$ and $\mathcal{Y}$ indeed induce the same growth-fragmentation. Roughly, this might fail to be true for the following reason. A cell $\mathcal{X}_u$ in the cell system $\mathcal{X}$ corresponds to an ancestral line in the new cell system $\mathcal{Y}$ (see Figure 1.7), and the birth times of the cells along this ancestral line of $\mathcal{Y}$ might accumulate to a finite limit $T > 0$. Then for every $t > T$, $\mathcal{X}_u(t)$ is not counted for in this new cell system. We hence need the following assumption, which indeed prevents this situation. Recall that f is a function that satisfies (1.2.1).

[Hη] There exists $\eta < 1$ such that for every $x > 0$ and every $s \geq 0$, there is

$$E_x \left[\sum_{t \geq 0} f\left(s + t, -\Delta X(t)\right) \right] \leq \eta f\left(s, x\right).$$

Note that [Hη] is not a consequence of [H] (since $\eta < 1$), and we did not request [Hη] for the homogeneous case since it holds automatically for every homogeneous process. We now state the main result.

Theorem 5.

Suppose that there exists a bifurcator of cell processes X and Y, and that X and Y satisfy [H] and [Hη]. Then the growth-fragmentations driven respectively by X and Y have the same finite-dimensional distributions.

In particular, we deduce by Theorem 5 that the law of a self-similar growth-fragmentation $\mathbf{X}^{(\alpha)\downarrow}$ is characterized by its cumulant κ and its index of self-similarity α.

Theorem 6.

Let $X^{(\alpha)}$ and $\tilde{X}^{(\tilde{\alpha})}$ be two self-similar processes with respective cumulants κ and $\tilde{\kappa}$, and $\mathbf{X}^{(\alpha)\downarrow}$ and $\tilde{\mathbf{X}}^{(\tilde{\alpha})\downarrow}$ be two Markovian growth-fragmentations driven respectively by $X^{(\alpha)}$ and $\tilde{X}^{(\tilde{\alpha})}$. Suppose that (1.2.6) holds for both κ and $\tilde{\kappa}$. Then the following statements are equivalent:

(i) $\kappa = \tilde{\kappa}$ *and* $\alpha = \tilde{\alpha}$;

(ii) $X^{(\alpha)}$ *and* $\tilde{X}^{(\tilde{\alpha})}$ *are the two respective marginal laws of a certain bifurcator;*

(iii) the self-similar growth-fragmentations $\mathbf{X}^{(\alpha)\downarrow}$ *and* $\tilde{\mathbf{X}}^{(\tilde{\alpha})\downarrow}$ *have the same law.*

Roughly speaking, "(i)$\Rightarrow$ (ii)" follows from the result in the homogeneous case and Lamperti's representation. Next, we can check that (1.2.6) ensures that [H] and [Hη] hold, so "(ii)$\Rightarrow$ (iii)" is a consequence of Theorem 5. Finally, "(iii)$\Rightarrow$ (i)" follows from the self-similarity **(P2)** and the many-to-one formula (1.2.7).

1.3 Ornstein-Uhlenbeck type growth-fragmentation processes

We present in this section the results developed in Chapter 4.

In both (pure) fragmentation and growth-fragmentation, the *self-similar* case has been emphasized. Here we introduce a new class of growth-fragmentation processes that possess a different scaling property. We name them *Ornstein-Uhlenbeck (OU) type growth-fragmentation processes*, as in such a particle system, very informally speaking, the growth of a particle is distributed according to the exponential of an OU type process $(Z(t), t \geq 0)$ driven by a Lévy process ξ:

$$Z(t) = \mathrm{e}^{-\theta t} Z(0) + \int_0^t \mathrm{e}^{-\theta(t-s)} d\xi(s), \quad t \geq 0. \tag{1.3.1}$$

where $\theta > 0$ and the integral is taken in the sense of a stochastic integral (a Lévy process is a semimartingale). If ξ is a Brownian motion, then Z is a classical Gaussian OU process.

Our initial motivation stems from a recent study of the destruction of an infinite recursive tree [10] (see also [67] for a related work on Bolthausen-Sznitman coalescent). Besides this motivation, this model may have other applications, as OU type processes are widely applied in various domains: in biology, they are used in a neuronal model with signal-dependent noise [60]; in finance, they are used in an option price model with stochastic volatility [7, 8], to name just a few.

Recall that $c_0^{\downarrow}$ is the space of decreasing null sequences endowed with the ℓ^∞-distance. Specifically, an OU type growth-fragmentation process is a $c_0^{\downarrow}$-valued càdlàg Markov process

$$\mathbf{X}^{\downarrow} := \left(\mathbf{X}^{\downarrow}(t) := (X_1(t), X_2(t), \ldots),\ t \geq 0 \right),$$

where $\mathbf{X}^{\downarrow}(t)$ is viewed as the decreasing sequence of the sizes of the particles alive at time t. For every $x \in [0, \infty)$, let $\mathbf{P}_x$ denote for the law of $\mathbf{X}^{\downarrow}$ with initial value $\mathbf{X}^{\downarrow}(0) = (x, 0, \cdots) \in c_0^{\downarrow}$. The process $\mathbf{X}^{\downarrow}$ further satisfies the branching property **(P1)** and

(P3) (The OU property) There exists a certain index $\theta > 0$, such that for every $x \in [0, \infty)$, the distribution of the rescaled process $(x^{\exp(-\theta t)} \mathbf{X}^{\downarrow}(t))_{t \geq 0}$ under $\mathbf{P}_1$ is $\mathbf{P}_x$.

This OU property has its root in the scaling property of an exponential OU type process (a direct consequence of (1.3.1)), which signifies that the mass of a fragment grows or decays gradually towards an equilibrium value. Note that, however, this OU type scaling property has no counterpart in (pure) fragmentations.

1.3.1 Preliminaries on Ornstein-Uhlenbeck type processes

We refer to [5] or Section 17 in [73] for background on Ornstein-Uhlenbeck (OU) type processes driven by Lévy processes.

Let $\theta > 0$ and ξ be a SNLP with characteristics (σ, c, Λ, k) and Laplace exponent Φ as in (1.2.2). Then for every $z \in \mathbb{R}$, the process Z defined as in (1.3.1) starting from $Z(0) = z$ is the path-wise unique solution of the stochastic integral equation

$$Z(t) = z + \xi(t) - \theta \int_0^t Z_s ds.$$

we say that the OU type process Z has characteristics $(\sigma, c, \Lambda, k, \theta)$, or simply (Φ, θ). Write P_z for the law of Z and E_z for the mathematical expectation under P_z. Then for every $t \geq 0$, there is

$$E_z[\exp(qZ(t))] = \exp\left(q\mathrm{e}^{-\theta t}z + \int_0^t \Phi(q\mathrm{e}^{-\theta s})ds\right), \quad q \geq 0. \tag{1.3.2}$$

Under a certain condition on the Lévy measure Λ of the Lévy process ξ, an OU type process driven by ξ converges in distribution to its stationary distribution.

Proposition 1.3.1 (Theorem 17.5 and 17.11 in [73]). *Let $\theta > 0$ and Λ satisfies*

$$\int_{(-\infty, -\log 2)} \log|z| \ \Lambda(dz) < \infty. \tag{1.3.3}$$

Then the OU type process Z possesses a unique stationary distribution Π, which is a probability measure with Laplace transform

$$\int_{\mathbb{R}} \mathrm{e}^{qy}\Pi(dy) = \exp\left(\int_0^\infty \Phi(\mathrm{e}^{-\theta s}q)ds\right), \quad q \geq 0.$$

In particular, for every $z \in \mathbb{R}$ and bounded and continuous function $g : \mathbb{R} \to \mathbb{R}$, there is

$$\lim_{t\to\infty} \mathbb{E}_z\left[g(Z(t))\right] = \int_{\mathbb{R}} g(y)\Pi(dy).$$

If (1.3.3) does not hold, then Z does not have any stationary distribution.

Remark 1.3.2. *The probability measure Π is self-decomposable, which means that if a random variable Y has law Π, then for every constant $r \in (0,1)$, there exists an independent random variable $Y^{(r)}$, such that $Y \overset{d}{=} rY + Y^{(r)}$. Conversely, every self-decomposable measure is the stationary distribution of a certain OU type process. See Definition 15.1 and Theorem 17.5 in [73] for details.*

A Poissonian construction of Z Denote the jump process by $\Delta Z(t) := Z(t) - Z(t-)$. We observe from (1.3.1) that $\Delta Z = \Delta\xi$, thus ΔZ is a Poisson point process with characteristic measure Λ on $(-\infty, 0)$, which is the Lévy measure of ξ. Note that Λ fulfills (1.2.3) and might be infinite. So we have a Poissonian construction of Z, which is similar to the well-known *Lévy-Itô decomposition* for Lévy processes, see e.g. [58] for the latter. The idea of this construction will later be used to build OU type growth-fragmentations.

Specifically, let $\sum_{i\in\mathbb{N}} \delta_{(z_i,t_i)}$ be a Poisson random measure with intensity $\Lambda(dz)\otimes dt$. For each $\epsilon>0$, since ξ has a finite number of jumps of sizes $<-\epsilon$, we can define a process

$$Z_{\epsilon,1}(t) := \sum_{0\le t_i\le t} \mathrm{e}^{-\theta(t-t_i)} \mathbf{1}_{\{\epsilon<|z_i|\}} z_i, \qquad t\ge 0.$$

Let $Z_{\epsilon,2}$ be an independent Gaussian OU process associated with the drifted Brownian motion

$$\sigma B_t + ct + t\int_{(-\infty,\epsilon)} (1-\mathrm{e}^{z})\,\Lambda(dz), \qquad t\ge 0. \tag{1.3.4}$$

For every $t\ge 0$, we define $Z(t)$ by the limit

$$Z(t) := \lim_{\epsilon\downarrow 0}(Z_{\epsilon,1}(t)+Z_{\epsilon,2}(t)),$$

where the convergence holds almost surely. Then Z is an OU type process with characteristics (Φ,θ). We stress that the extra drift coefficient in (1.3.4) aims at compensating $Z_{\epsilon,1}$. If we do not put this term and $\int_{(-1,0)}|z|\Lambda(dz)=\infty$, then as $\epsilon\downarrow 0$, the accumulation of small jumps would make the limit process explode (i.e. jump to $-\infty$) instantaneously.

1.3.2 The construction of OU type growth-fragmentations

We now present the construction of OU type growth-fragmentations, which is inspired by [19]. At the heart of this approach lies the fact that the logarithm of a homogeneous fragmentation is a (continuous time) branching random walk, see [25]. Similarly, we shall build *an OU type branching Markov process*, which can be viewed as the logarithm of an OU type growth-fragmentation.

Let us start with the case when branching occurs with a finite intensity, which shall be specified as an *OU type branching Markov chain*. Its dynamics are characterized by a quadruple (σ^2,c,μ,θ), where $\sigma^2\ge 0$, $c\in\mathbb{R}$, $\theta>0$ and μ is a sigma-finite measure on the space

$$\mathcal{R} := \left\{\mathbf{r}=(r_1,r_2,\dots)\ :\ 0\ge r_1\ge r_2\ge\dots\ge-\infty, \sum_{i=1}^{\infty}\mathrm{e}^{r_i}\le 1\right\},$$

that satisfies

$$\int_{\mathcal{R}}(1-\mathrm{e}^{r_1})^2\mu(d\mathbf{r})<\infty. \tag{1.3.5}$$

Definition 1.

Suppose that (1.3.5) holds and further that

$$\mu(\mathcal{R}\setminus\mathcal{R}_1)<\infty,\ \ \textit{where } \mathcal{R}_1 := \{\mathbf{r}\in\mathcal{R}\ :\ r_1>-\infty, r_2=r_3=\dots=-\infty\}.$$

We construct a particle system by the following description.

- *At the initial time, a single particle is located at* 0.
- *(The spatial displacement) The position of each particle evolves according to an OU type process Z with characteristics (ψ, θ), where*

$$\psi(q) := \frac{1}{2}\sigma^2 q^2 + \Big(c + \int_{\mathcal{R}\setminus\mathcal{R}_1} (1 - \mathrm{e}^{r_1})\mu(d\mathbf{r})\Big) q + \int_{\mathcal{R}_1} (\mathrm{e}^{qr_1} - 1 + q(1 - \mathrm{e}^{r_1}))\, \mu(d\mathbf{r}), \quad q \geq 0. \tag{1.3.6}$$

- *(The branching events) For every $\mathbf{r} = (r_1, r_2, \dots) \in \mathcal{R}$, a particle at position $y \in \mathbb{R}$ splits into a cloud of particles at $y + \mathbf{r}$ with rate $\mu|_{\mathcal{R}\setminus\mathcal{R}_1}(d\mathbf{r})$, and the particle born at position $y + r_i$ evolves according to the law of Z starting from $Z(0) = y + r_i$.*

For every $t \geq 0$, write $\mathbf{Z}(t)$ for the multiset of the positions of the particles alive at time t, then the process $\mathbf{Z} := (\mathbf{Z}(t))_{t\geq 0}$ is called an **OU type branching Markov chain** *with characteristics (σ, c, μ, θ).*

The extra drift term in (1.3.6) corresponds to the compensation of the jumps of the particles caused by the branching events, playing a similar role to that in (1.3.4). Thanks to this properly chosen drift coefficient, we obtain a key embedding property as follows. For every $\ell \geq 0$, cut an OU type branching Markov chain at level ℓ, by keeping at each dislocation the child particle which is the closest to the parent, and by suppressing every other child particle, as well as all its progeny, if and only if its distance to the position of the parent at death is larger than or equal to ℓ. This operation yields a truncated process $\mathbf{Z}^{(\ell)}$, which is an OU type branching Markov chain with characteristics $(\sigma^2, c, \mu^{(\ell)}, \theta)$, where $\mu^{(\ell)}$ is the image of μ by a map $\mathcal{R} \to \mathcal{R}$ given by $(r_1, r_2, r_3, \dots) \mapsto (r_1, r_2^{(\ell)}, r_3^{(\ell)}, \dots)$, where $r^{(\ell)} = r$ if $r > -\ell$ and $r^{(\ell)} = -\infty$ if $r \leq -\ell$.

We next drop the requirement $\mu(\mathcal{R} \setminus \mathcal{R}_1) < \infty$ and only suppose that (1.3.5) holds, such that the branching might occur with an infinite intensity. By the consistency of the truncation and Kolmogorov's extension theorem, we build a family of $(\mathbf{Z}^\ell)_{\ell\geq 0}$ on the same probability space, such that each $\mathbf{Z}^\ell$ is an OU type branching Markov chain with characteristics $(\sigma, c, \mu^{(\ell)}, \theta)$. Further, consider for every $\ell' > \ell$ the process $(\mathbf{Z}^\ell)^{(\ell')}$ derived from $\mathbf{Z}^\ell$ by truncating at level ℓ', then there is $(\mathbf{Z}^\ell)^{(\ell')} = \mathbf{Z}^{\ell'}$.

Definition 2.

Suppose that (1.3.5) holds. In the notation above, we define

$$\mathbf{Z}(t) := \lim_{\ell\to\infty} \uparrow \mathbf{Z}^\ell(t), \quad t \geq 0.$$

Then the process $\mathbf{Z}$ *is called an* **OU type branching Markov process** *with characteristics* (σ, c, μ, θ).

We can naturally associate a growth-fragmentation process with an OU type branching Markov process. Recall that the mass-partition space is denoted by $\mathcal{S}$. Let ν be a sigma-finite measure on $\mathcal{S}$ that satisfies

$$\int_{\mathcal{S}} (1 - s_1)^2 \nu(d\mathbf{s}) < \infty. \tag{1.3.7}$$

We stress that (1.3.7) is a weaker condition than the condition (1.1.1) for a pure fragmentation. Let μ be the image of measure ν by the map $\mathbf{s} \mapsto (\log(s_1), \log(s_2), \ldots) \in \mathcal{R}$, then μ satisfies (1.3.5).

Definition 3.

Let $\mathbf{Z}$ *be an OU type branching Markov process with characteristics* (σ, c, μ, θ). *The process*

$$\mathbf{X}(t) := \{\!\{\exp(z) : \ z \in \mathbf{Z}(t)\}\!\}, \quad t \geq 0$$

is called an **OU type growth-fragmentation process** *with characteristics* (σ, c, ν, θ).

Informally speaking, $\sigma \geq 0$ describes the fluctuations of the size, the constant $c \in \mathbb{R}$ represents the deterministic dilation (resp. erosion) coefficient when $c > 0$ (resp. $c < 0$). The measure ν is called the *dislocation measure*. Roughly speaking, for every $\mathbf{s} \in \mathcal{S}$, a fragment of size $x > 0$ splits into a sequence of fragments $x\mathbf{s}$ at rate $\nu(d\mathbf{s})$. The constant $\theta > 0$ characterizes the speed at which the size of a fragment evolves towards the centred value (normalized to be 1).

Remark 1.3.3. *An OU type process* (1.3.1) *is also well-defined when* $\theta \leq 0$, *and the above construction of OU type growth-fragmentations still works for this case. Details are given in Chapter 4. However, Proposition 1.3.1 does not hold for the case* $\theta \leq 0$. *When* $\theta = 0$, *an OU type growth-fragmentation with characteristics* $(\sigma, c, \nu, 0)$ *coincides with a compensated fragmentation process with characteristics* (σ, c, ν) *in the sense of Definition 3 in [19], which can be viewed as a general homogeneous growth-fragmentation (that fulfills* **(P1)** *and* **(P2)** *with* $\alpha = 0$*).*

The crucial property that needs to be verified is that the process $\mathbf{X}$ does not explode. In analogy to (1.2.4), we introduce the *cumulant* $\kappa : [0, \infty) \to (\infty, \infty]$ by

$$\kappa(q) := \frac{1}{2}\sigma^2 q^2 + cq + \int_{\mathcal{R}} \Big(\sum_{i=1}^{\infty} s_i^q - 1 + q(1 - s_1)\Big)\nu(d\mathbf{s}), \quad q \geq 0. \tag{1.3.8}$$

The function κ is convex and it follows from (1.3.7) that $\kappa(q) < \infty$ for $q \in [2, \infty)$. We obtain the counterpart of (1.2.5) or (1.2.7) as follows.

Theorem 7.

For every $t \geq 0$, the elements of $\mathbf{X}(t)$ can be ranked in decreasing order, which yields a decreasing sequence

$$\mathbf{X}^{\downarrow}(t) := (X_1(t), X_2(t), \cdots).$$

Further, for every $q \geq 2\mathrm{e}^{\theta t}$ we have

$$\mathbb{E}\left[\sum_{i=1}^{\infty} X_i(t)^q\right] = \exp\left(\int_0^t \kappa(q\mathrm{e}^{-\theta s})ds\right). \tag{1.3.9}$$

The identity (1.3.9) has a similar form as (1.3.2). With no risk of confusion, the process $\mathbf{X}^{\downarrow}$ is also called an OU type growth-fragmentation. One can essentially deduce from Theorem 7 that $\mathbf{X}^{\downarrow}$ indeed fulfills the desired properties:

Proposition 4.

The process $\mathbf{X}^{\downarrow}$ is Markovian, fulfills the branching and OU type scaling properties, and has a càdlàg version in $c_o^{\downarrow}$.

As a direct consequence, for every $p \geq 2$, the process

$$\exp\Big(-\int_0^t \kappa(p\mathrm{e}^{\theta s})ds\Big)\sum_{i=1}^{\infty} X_i(t)^{p\mathrm{e}^{\theta t}}, \qquad t \geq 0$$

is a non-negative martingale, which should be compared with the famous *additive martingales* in context of fragmentations [25] or branching random walks [28].

1.3.3 Different aspects of OU type growth-fragmentations

The OU type growth-fragmentation $\mathbf{X}^{\downarrow}$ is the stochastic counterpart of a (deterministic) growth-fragmentation equation in the following sense.

Proposition 5.

For every $t \geq 0$, define a Radon measure $\rho(t)$ such that

$$\langle \rho(t), f \rangle := \mathbb{E}\Big[\sum_{i=1}^{\infty} f(X_i(t))\Big] \qquad \text{for all } f \in C_c^\infty(0,\infty).$$

Then $(\rho(t), t \geq 0)$ is a solution to the growth-fragmentation equation

$$\frac{\partial}{\partial t}\langle \rho(t), f \rangle = \langle \rho(t), \mathcal{L}f \rangle,$$

where

$$\begin{aligned} \mathcal{L}f := \;& \frac{1}{2}\sigma^2 x^2 f''(x) + \big(c + \frac{1}{2}\sigma^2 - \theta \log x\big) x f'(x) \\ & + \int_{\mathcal{S}} \Big(\sum_{i=1}^{\infty} f(xs_i) - f(x) + xf'(x)(1-s_1)\Big)\nu(d\mathbf{s}). \end{aligned} \tag{1.3.10}$$

See [48] for fragmentation equations related to self-similar fragmentations and [27, 21] for growth-fragmentation equations related to self-similar growth-fragmentations. Different from their proofs which all rely on a spine technique, we use an idea of discretization by first dealing with the truncated growth-fragmentations (as in Definition 2) and then passing to the limit.

OU type growth-fragmentations are connected with Markovian growth-fragmentations presented in Section 1.2.1. Recall that the cumulant κ of $\mathbf{X}^{\downarrow}$ is given by (1.3.8).

Proposition 6.

Suppose that $\mathbf{X}^{\downarrow}$ is binary, i.e. its dislocation measure ν has support on

$$\{\mathbf{s} \in \mathcal{S} \; : \; s_1 + s_2 = 1, s_3 = s_4 = \ldots = 0\} \bigcup \{(0,0,\ldots)\}.$$

Then the law of $\mathbf{X}^{\downarrow}$ is characterized by (κ, θ). Further, for each OU type process Z with characteristics (Φ, θ), if the cumulant of Z defined by (1.2.4) is also κ, then the Markovian growth-fragmentation associated with $\exp(Z)$ has the same law as $\mathbf{X}^{\downarrow}$.

We finally lift from [10] an example of an OU type growth-fragmentation. Let Z^R be an OU type process with characteristics $(0, -c_\gamma + 1, \Lambda, 0, 1)$, where $c_\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant and the Lévy measure Λ has density

$$\Lambda(dz) = \mathrm{e}^z(1-\mathrm{e}^z)^{-2}dz, \quad z \in (-\infty, 0).$$

Then the Markovian growth-fragmentation $\mathbf{X}^{R\downarrow}$ associated with $\exp(Z^R)$ is a binary OU type growth-fragmentation with characteristics $(\kappa_R, 1)$ (in the sense of Proposition 6), where

$$\kappa_R(q) = q\psi(q+1) + (q-1)^{-1}, \quad q > 1,$$

where ψ denotes the digamma function, that is the logarithmic derivative of the gamma function. In particular, for every $t \geq 0$ and $q > \mathrm{e}^t$, it follows from (1.3.9) that

$$\mathbb{E}\left[\sum_{i=1}^{\infty} X_i^R(t)^q\right] = \frac{q-1}{\mathrm{e}^{-t}q - 1} \frac{\Gamma(q)}{\Gamma(\mathrm{e}^{-t}q)}.$$

This OU type growth-fragmentation appears naturally in the destruction of an infinite recursive tree by removing each edge after an independent exponential time, see [10] and Chapter 4 for details.

1.3.4 A law of large numbers

Consider an OU type growth-fragmentation $\mathbf{X}^{\downarrow} = (X_1(t), X_2(t), \ldots)_{t\geq 0}$ with cumulant κ. Under certain conditions, we prove a law of large numbers which, roughly speaking, states that the average of the sizes of the fragments converges to a limit distribution as time tends to infinity.

Let us first present the assumptions that we need. For every $t \geq 0$, the number of fragments at time t is denoted by

$$N(t) := \sum_{i=1}^{\infty} \mathbf{1}_{\{X_i(t) \neq 0\}}.$$

We suppose $\kappa(0) < \infty$, then the process $(N(t), t \geq 0)$ is simply a *branching process*, see e.g. [6]. We further suppose that N is supercritical (i.e. $\mathbb{P}\left(\lim_{t\to\infty} N(t) = 0\right) < 1$) and that there exists $\gamma \in (1, 2]$ such that the non-negative martingale

$$M(0,t) := \exp\left(-\kappa(0)t\right) N(t), \quad t \geq 0$$

converges to $M(0,\infty)$ almost surely and in $L^\gamma(\mathbb{P})$. The necessary and sufficient conditions for these assumptions are well-known from basic properties of branching processes. See details in Chapter 4.

We next describe the limit distribution. There exists a SNLP with Laplace exponent

$$\Phi_0(q) := \kappa(q) - \kappa(0), \quad q \geq 0.$$

Then it follows from Proposition 1.3.1 that

$$\int_{\mathcal{S}} \sum_{i=1}^{\infty} \mathbf{1}_{\left\{s_i < \frac{1}{2}\right\}} \log(|\log s_i|) d\nu(d\mathbf{s}) < \infty$$

is the sufficient and necessary condition for an OU type process with characteristics (Φ_0, θ) to possess a unique stationary distribution, denoted by Π_0. Recall from Remark 1.3.2 that Π_0 is a self-decomposable probability measure. Let $\tilde{\Pi}_0$ be the image of Π_0 by the map $y \mapsto \mathrm{e}^y$.

Theorem 8.

Under the above assumptions, for every continuous function f on $(0, \infty)$ with compact support, we have

$$\lim_{t\to\infty} \exp^{-\kappa(0)t} \sum_{i=1}^{\infty} f(X_i(t)) = \langle \tilde{\Pi}_0, f \rangle M(0, \infty) \qquad \text{in } L^\gamma(\mathbb{P}). \tag{1.3.11}$$

As a consequence, conditionally on non-extinction, there is

$$\lim_{t\to\infty} N(t)^{-1} \sum_{i=1}^{\infty} f(X_i(t)) = \langle \tilde{\Pi}_0, f \rangle \quad \text{in probability.}$$

Roughly speaking, this law of large numbers holds since the size of a "typical" fragment evolves according to the exponential of an OU type process with characteristics (Φ_0, θ), and two "typical" fragments are "almost" independent after long time. Theorem 8 should be compared with the law of large numbers for branching diffusions [43], the asymptotic behavior of self-similar growth-fragmentations (in particular self-similar pure fragmentations) [23, 38], and the convergence results of Crump-Mode-Jagers branching processes [68, 50].

Taking expectation to (1.3.11), we deduce the long time asymptotic for the solution $(\rho(t), t \geq 0)$ to the growth-fragmentation equation in Proposition 5. See [66] and references therein for similar estimates.

Corollary 1.

Under the assumptions of Theorem 8, as $t \to \infty$, we have weak convergence

$$\mathrm{e}^{-\kappa(0)t} \rho(t) \Longrightarrow \tilde{\Pi}_0.$$

Further, $\tilde{\Pi}_0$ is a solution to the stationary equation:

$$\langle \tilde{\Pi}_0, \mathcal{L} f \rangle = \kappa(0) f, \qquad \text{for every } f \in \mathcal{C}_c^\infty(0, \infty),$$

where $\mathcal{L}$ is as in (1.3.10).

Chapter 2

Large Triangles In The Brownian Triangulation And Fragmentation Processes

This chapter is mainly based on [75], and we add a new section on large dislocations in dissipative fragmentations, Section 2.5.

The Brownian triangulation is a random compact subset of the unit disk introduced by Aldous. For $\epsilon > 0$, let $N(\epsilon)$ be the number of triangles whose sizes (measured in different ways) are greater than ϵ in the Brownian triangulation. We determine the asymptotic behavior of $N(\epsilon)$ as $\epsilon \to 0$.

To obtain this result, a novel concept of "large" dislocations in fragmentations has been proposed. We develop an approach to study the number of large dislocations which is widely applicable to general self-similar fragmentation processes. This technique enables us to study $N(\epsilon)$ because of a bijection between the triangles in the Brownian triangulation and the dislocations of a certain self-similar fragmentation process.

Our method also provides a new way to obtain the law of the length of the longest chord in the Brownian triangulation. We further extend our results to the more general class of geodesic stable laminations introduced by Kortchemski.

Note: for simplicity, throughout this chapter we denote a process with values in the mass-partition space $\mathcal{S}$ by $\mathbf{X}$ (instead of $\mathbf{X}^{\downarrow}$ which is used in the other chapters).

2.1 Introduction

For $n \in \mathbb{N}$, let P_n be the polygon formed by the n roots of unity. A *triangulation of* P_n is the union of its sides and $(n-3)$ non-crossing (except at the endpoints) diagonals, thus dividing P_n

into $(n-2)$ triangles. A *uniform triangulation* $\mathcal{T}_n$ *of* P_n is a triangulation chosen uniformly at random from the set of all the different triangulations of P_n. In [3], Aldous regarded $\mathcal{T}_n$ as a random compact subset of the closed unit disk $\mathbb{D} \subset \mathbb{R}^2$ and showed that, as n tends to infinity, $\mathcal{T}_n$ converges to a limit random compact set $\mathcal{B}$ in distribution for the Hausdorff metric. Figure 2.1 shows a sample of $\mathcal{B}$.

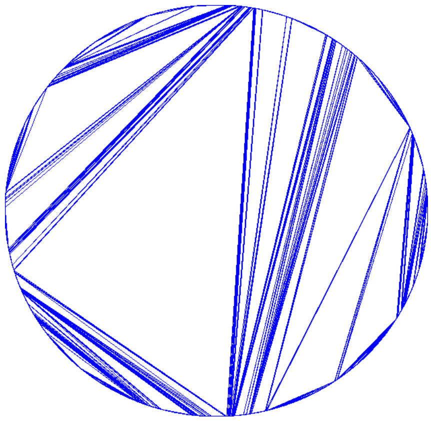

Figure 2.1: **A sample of the Brownian triangulation**

It turns out that $\mathcal{L}_{\mathbf{e}}$ is a random *triangulation of the disk*, in the sense that it is a random closed subset of $\mathbb{D}$, whose complement $\mathbb{D}\backslash\mathcal{B}$ is a union of open triangles with vertices on the unit circle $\partial\mathbb{D}$. Aldous called $\mathcal{L}_{\mathbf{e}}$ the *Brownian triangulation* since it can be encoded by a normalized Brownian excursion $e = (e_s, s \in [0,1])$ as follows. Parameterize $\partial\mathbb{D}$ by $(\mathrm{e}^{i2\pi s}, s \in [0,1))$, and write $[\mathrm{e}^{i2\pi s}, \mathrm{e}^{i2\pi t}]$ for the chord connecting $\mathrm{e}^{i2\pi s}, \mathrm{e}^{i2\pi t} \in \partial\mathbb{D}$. Then almost surely

$$\mathcal{B} = \bigcup_{s \overset{e}{\sim} t, s,t \in [0,1)} [\mathrm{e}^{i2\pi s}, \mathrm{e}^{i2\pi t}],$$

where $s \overset{e}{\sim} t$ if and only if $e(s) = e(t) = \min_{r \in [s \wedge t, s \vee t]} e(r)$. See [3] for details.

The Brownian triangulation draws our attention because of its importance in many aspects. The Brownian triangulation is universal, as it is the limit of various random non-crossing configurations (collections of non-crossing diagonals) of P_n [35]. The Brownian triangulation is also closely related to the Brownian Continuum random tree (CRT) [2, 3] and the Brownian map [61]. Finally, the Brownian triangulation has provoked the study of other random triangulations, such as random recursive triangulations [36] and the Markovian hyperbolic triangulation [37].

By definition, a *triangle*, or *face*, of $\mathcal{L}_{\mathbf{e}}$ is a connected component of $\mathbb{D}\backslash\mathcal{L}_{\mathbf{e}}$. In the present work, we are mainly interested in the number of "large" triangles in $\mathcal{L}_{\mathbf{e}}$. Clearly there are various ways of measuring the size of a triangle. Here we are concerned with two different ways, specifically the length of the shortest edge and the area.

Let us now present a special case of our results. Recall that e is the normalized Brownian excursion that encodes $\mathcal{B}$. We define a family of random open sets

$$\Theta_e(t) := \{s \in (0,1) : e(s) > t\}, \quad t \geq 0. \tag{2.1.1}$$

For every $t \geq 0$, write $\Theta_e(t) = \bigcup_{i\in\mathbb{N}} I_i(t)$, where $(I_i(t), i \in \mathbb{N})$ are the connected components of $\Theta_e(t)$. Hence $(I_i(t), i \in \mathbb{N})$ are disjoint open intervals, possibly empty. We denote the length of an interval I by $|I|$.

Theorem 2.1.1.

1. *For every $\epsilon > 0$, let $N'(\epsilon)$ be the number of triangles in $\mathcal{B}$ whose edges have lengths greater than ϵ. There is*
$$\lim_{\epsilon\to 0} \epsilon N'(\epsilon) = 2 \quad \text{in } L^2(\mathbb{P}).$$

2. *Let $N''(\epsilon)$ be the number of triangles in $\mathcal{B}$ whose Euclidean area is larger than $\epsilon > 0$. There is*
$$\lim_{\epsilon\to 0} \epsilon^{\frac{1}{2}} N''(\epsilon) = 4 \int_0^\infty \sum_{i=1}^\infty \sin(\pi |I_i(s)|) ds \quad \text{in } L^2(\mathbb{P}).$$

We note that in the first case the limit is a constant, while in the second case the limit is a random variable. It turns out that this surprising phenomenon is an instance of a general phase transition revealed in Theorem 2.2.6 below. To justify that the random variable $\int_0^\infty \sum_{i=1}^\infty \sin(\pi|I_i(s)|)ds$ is indeed square integrable, let us compare it with the *Brownian excursion area* $\mathcal{A}_e$,

$$\mathcal{A}_e := \int_0^\infty \sum_{i=1}^\infty |I_i(t)| dt = \int_0^1 e(s) ds.$$

It is known that $\mathbb{E}\left[\mathcal{A}_e^k\right] < \infty$ for every $k \in \mathbb{N}$, see [51]. Noticing that

$$\int_0^\infty \sum_{i=1}^\infty \sin(\pi|I_i(s)|) ds \leq \pi \mathcal{A}_e,$$

we see that it is indeed square integrable.

It has been proved in [3] that for $0 \leq x_1 \leq x_2 \leq x_3 \leq 1$, the expected value of the number of the triangles whose vertices are at position $(e^{i2\pi x_1}, e^{i2\pi x_2}, e^{i2\pi x_3})$ has density

$$\frac{1}{4\pi}(x_2 - x_1)^{-\frac{3}{2}}(x_3 - x_2)^{-\frac{3}{2}}(1 + x_1 - x_3)^{-\frac{3}{2}} dx_1 dx_2 dx_3, \quad 0 \leq x_1 \leq x_2 \leq x_3 \leq 1.$$

By integrating the density function, we may deduce that

$$\lim_{\epsilon\to 0} \epsilon \mathbb{E}\left[N'(\epsilon)\right] = 2, \quad \text{and} \quad \lim_{\epsilon\to 0} \epsilon^{\frac{1}{2}} \mathbb{E}\left[N''(\epsilon)\right] = \frac{\sqrt{2}\pi}{2} J_1(\frac{\pi}{2}),$$

where J_1 is the Bessel function of the first kind, with $J_1(\frac{\pi}{2}) \approx 0.5668$. However, this result is weaker than our convergence in $L^2(\mathbb{P})$ for random variables.

Theorem 2.1.1 is proved in Section 2.3. Our approach to tackle this problem is through a connection with fragmentation processes. It has been proved by Bertoin [15] that the process Θ_e given by (2.1.1) is an example of a *self-similar interval-partition fragmentation with index* $-\frac{1}{2}$ (see Section 2.2.1 for background). Roughly speaking, The process Θ_e describes how the interval $(0,1)$ splits into smaller intervals as time grows. For $s > t$, $\Theta_e(s)$ is obtained from $\Theta_e(t)$ by breaking randomly into pieces each component of $\Theta_e(t)$ according to a law that only depends on the length of this component, and independently of the others. We will specify this law in Section 2.3. An interval splitting event is called a *dislocation*. We point out that in each dislocation of Θ_e, an interval $I \subset (0,1)$ of length $|I|$ must split into two pieces (I_1, I_2) with $|I_1| + |I_2| = |I|$. Such a dislocation is marked by $(|I|, (|I_1|/|I|, |I_2|/|I|)) \in (0,1] \times \Delta$, where $\Delta := \{(s_1, s_2) \in [0,1]^2 : s_1 + s_2 = 1, s_1 \geq s_2\}$.

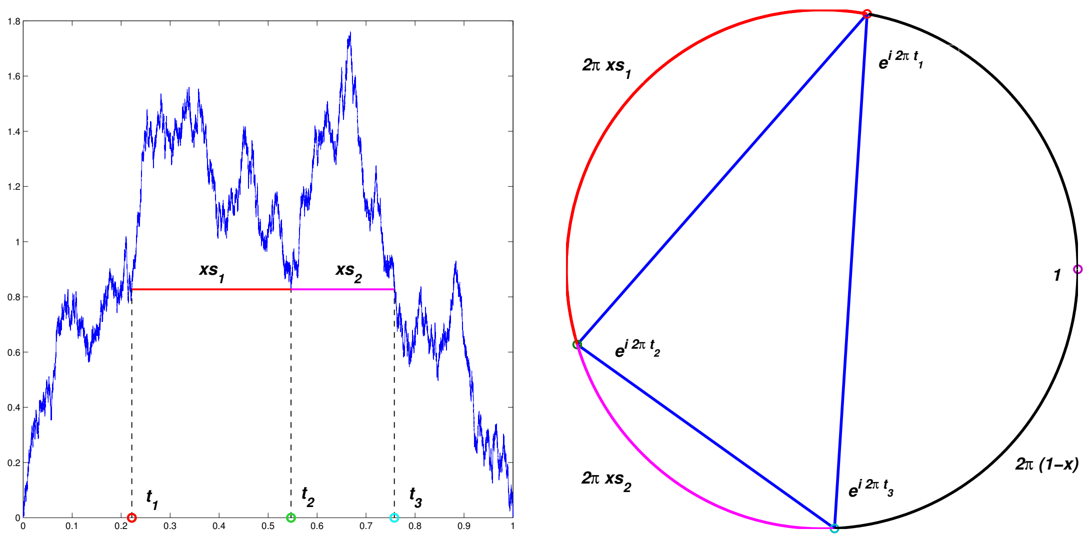

Figure 2.2: **The correspondence between dislocations and triangles.** The local minimum t_2 of the Brownian excursion e on the left induces a dislocation of Θ_e, which corresponds to the triangle in $\mathcal{L}_\mathbf{e}$ on the right. In this dislocation the interval (t_1, t_3) of length $x = t_3 - t_1$ produces two intervals (t_1, t_2) and (t_2, t_3). Set $s_1 = \max(t_2 - t_1, t_3 - t_2)/x$ and $s_2 = 1 - s_1$, then this dislocation is marked by $(x, (s_1, s_2))$. Since $t_1 \overset{e}{\sim} t_2 \overset{e}{\sim} t_3$, the chords $[\mathrm{e}^{i2\pi t_1}, \mathrm{e}^{i2\pi t_2}]$, $[\mathrm{e}^{i2\pi t_2}, \mathrm{e}^{i2\pi t_3}]$ and $[\mathrm{e}^{i2\pi t_3}, \mathrm{e}^{i2\pi t_1}]$ are included in $\mathcal{L}_\mathbf{e}$, and they form a triangle. Hence this dislocation in Θ_e marked by $(x, (s_1, s_2))$ corresponds to the triangle in $\mathcal{B}$ whose vertices divide the circle into three arcs of lengths $(2\pi(1-x), 2\pi x s_1, 2\pi x s_2)$.

The following observation plays a key role in this work.

Proposition 2.1.2. *There is a bijection between the faces in $\mathcal{B}$ and the dislocations in Θ_e. If a dislocation in Θ_e is marked by $(x, (s_1, s_2)) \in (0,1] \times \Delta$, then the corresponding triangle in $\mathcal{B}$ has edges of lengths $(2\sin(\pi x), 2\sin(\pi x s_1), 2\sin(\pi x s_2))$.*

A formal proof of Proposition 2.1.2 is given in Section 2.3. This correspondence is illustrated in Figure 2.2. This bijection should be clear since the faces in $\mathcal{B}$ and the dislocations in Θ_e are both in bijection with the local minima of e. The second statement is simply obtained by basic geometry. By this bijection, if a triangle in $\mathcal{L}_{\mathbf{e}}$ corresponds to a dislocation in Θ_e marked by $(x,(s_1,s_2)) \in (0,1] \times \Delta$, then the length of its shortest edge is

$$\psi'(x,(s_1,s_2)) := \min(2\sin(\pi x), 2\sin(\pi x s_1), 2\sin(\pi x s_2)). \tag{2.1.2}$$

Observing that the angle between the edge of length $2\sin(\pi x s_1)$ and the edge of length $2\sin(\pi x s_2)$ is $\pi(1-x)$, we find that the area of this triangle is

$$\psi''(x,(s_1,s_2)) := 2\sin(\pi x s_1)\sin(\pi x s_2)\sin(\pi x). \tag{2.1.3}$$

Hence with our fragmentation point of view, $N'(\epsilon)$ is the number of dislocations in Θ_e whose marks satisfy $\psi'(x,(s_1,s_2)) > \epsilon$. A similar statement holds for $N''(\epsilon)$. In Section 2.2.4, we introduce the notion of *large dislocations* that generalizes these families of dislocations. We study the number of large dislocations in the context of a general self-similar fragmentation and obtain Theorem 2.2.6 below, which leads to the final proof of Theorem 2.1.1. We see that a phase transition appears in Theorem 2.2.6, which explains the different limits in the two parts of Theorem 2.1.1. Our results on large dislocations are quite general which also enable us to answer the following two questions.

The first one is to study a generalization of the Brownian triangulation, the (geodesic) stable laminations of the disk introduced by Kortchemski [54]. For $\beta \in (1,2]$, the *β-stable lamination* is a random collection of non-crossing chords of the disk, which coincides with the Brownian triangulation when $\beta = 2$, and is encoded by the normalized excursion of β-strictly stable Lévy process when $\beta \in (1,2)$. For $\beta \in (1,2)$, we find a bijection between the faces (which are not triangles) in the β-stable lamination and the dislocations in a certain self-similar fragmentation, which enables us to study the number of large faces in the β-stable lamination.

The second question is to determine the law of the length of the longest chord. For the Brownian triangulation, this has been calculated in [3] by using discrete approximation by T_n; for the stable laminations, it is an open question due to Kortchemski, which is also mentioned in [33]. Noticing that the longest chord is an edge of the *centroid*, the (almost surely) unique face that contains the origin, we will answer this question by exploring the dislocation associated with the centroid.

In short, we develop a study of the number of large dislocations in self-similar fragmentations and apply our results to estimate the number of faces in the Brownian triangulation and stable laminations. Our method also opens the way to study a number of other interesting problems. To mention just a few, we may consider the role of large dislocations in random recursive triangulations [36], self-similar trees [49] and quadtrees [32].

The rest of this Chapter is organized as follows. In Section 2.2, we study the number of large dislocations in self-similar fragmentations. In Section 2.3, we prove Theorem 2.1.1 and find the law of the length of the longest chord in the Brownian triangulation. In Section 2.4, we investigate the large faces and the longest chord in the stable laminations. In Section 2.6, we complete the proofs of Lemma 2.2.12 and Lemma 2.2.13.

2.2 Large dislocations in a self-similar fragmentation

In this section we study the number of large dislocations in self-similar fragmentations. The main result, Theorem 2.2.6, is stated and proved in Section 2.2.4. Before that, we briefly review some basic facts about self-similar fragmentations in Section 2.2.1, and, in preparation for proving Theorem 2.2.6, we explain how to change the index of self-similarity in Section 2.2.2 and discuss the tagged fragment in Section 2.2.3.

2.2.1 Background on self-similar fragmentations

We refer to [11, 14, 15] for the general framework of self-similar fragmentations. Here we only give a short presentation. *A self-similar mass fragmentation with index of self-similarity* $\alpha \in \mathbb{R}$ is a càdlàg Markov process $\mathbf{X}^{(\alpha)} = \Big((X_1^{(\alpha)}(t), X_2^{(\alpha)}(t), \ldots),\ t \geq 0\Big)$ taking values in

$$\mathcal{S} := \Big\{\mathbf{s} = (s_1, s_2, s_3, \cdots) : 1 \geq s_1 \geq s_2 \geq \cdots \geq 0,\ \textit{and}\ \sum_{i=1}^{\infty} s_i \leq 1\Big\},$$

which satisfies the *branching* and *scaling* properties. The branching property means that for every sequence $\mathbf{x} = (x_1, x_2, \cdots) \in \mathcal{S}$ and every $t \geq 0$, the distribution of $\mathbf{X}^{(\alpha)}$ given $\mathbf{X}(0) = \mathbf{x}$ is the same as the union of the masses, arranged in the decreasing order, of a sequence of independent fragmentations $(\mathbf{X}^i)_{i\geq 1}$, where each $\mathbf{X}^i$ has distribution $\mathbb{P}_{x_i}$, the law of $\mathbf{X}^{(\alpha)}$ that starts from the state $x_i := (x_i, 0, \cdots) \in \mathcal{S}$. The scaling property means that for $x \in [0, 1]$, the distribution of the re-scaled process $(x\mathbf{X}^{(\alpha)}(x^{\alpha}t))_{t\geq 0}$ under $\mathbb{P}_1$ is $\mathbb{P}_x$.

For simplicity, throughout the rest of this paper we will implicitly suppose that any fragmentation starts from a single fragment with unit mass, and we will work under $\mathbb{P} := \mathbb{P}_1$.

A self-similar fragmentation is characterized by a triple (α, c, ν): $\alpha \in \mathbb{R}$ is the index of self-similarity; the non-negative real constant c is the *erosion rate*, which describes the speed at which the fragments melt continuously; the σ-finite measure ν on $\mathcal{S}$ verifying

$$\nu(\{(1, 0, \cdots)\}) = 0,\ \text{and}\ \int_{\mathcal{S}} (1 - s_1)\nu(d\mathbf{s}) < \infty$$

is the *dislocation measure*, which describes the statistics of the smaller pieces generated in a dislocation. For $\mathbf{s} = (s_1, s_2, \cdots) \in \mathcal{S}$, a fragment of mass x splits into masses $(xs_1, xs_2, \cdots)$ at rate $x^\alpha \nu(d\mathbf{s})$. We say a fragmentation is *conservative* if its dislocation measure satisfies

$$\nu\left(\mathbf{s} \in \mathcal{S} : \sum_{i=1}^{\infty} s_i < 1\right) = 0.$$

Otherwise it is *dissipative*.

A parallel notion is the *self-similar interval-partition fragmentations*, which was mentioned in the introduction. An interval-partition fragmentation $\Theta = (\Theta(t), t \geq 0)$ studies how the interval $(0,1)$ splits into smaller open intervals as time grows. If the existing intervals at $t > 0$ form a sequence $(I_1(t), I_2(t), \cdots)$, arranged in the decreasing order of length, then the state of the interval-partition fragmentation at t is their union $\Theta(t) = \bigcup_{i\in\mathbb{N}} I_i(t)$. Clearly $(\Theta_t, t \geq 0)$ is a family of nested open subsets of $(0,1)$. We observe that an interval-partition fragmentation naturally yields a mass fragmentation, specifically the length sequence process $(|I_1(t)|, |I_2(t)|, \cdots)_{t\geq 0}$. Therefore, we call Θ a self-similar interval-partition fragmentation if Θ is associated with a self-similar mass fragmentation.

2.2.2 Changing the index of self-similarity

A self-similar fragmentation process with index of self-similarity zero is a *homogeneous fragmentation process*. For any self-similar fragmentation $\mathbf{X}^{(\alpha)}$ with no erosion and index of self-similarity $\alpha \in \mathbb{R}$, we are able to change the index α to 0 by the following transformation introduced in [15]. Let $\Theta^{(\alpha)}$ be an interval fragmentation whose associated mass fragmentation is $\mathbf{X}^{(\alpha)}$ as in Section 2.2.1. For $x \in (0,1)$ and $t \geq 0$, if $x \in \Theta^{(\alpha)}(t)$, then let $I_x^{(\alpha)}(t)$ be the interval component of $\Theta^{(\alpha)}(t)$ that contains x at time t; if $x \notin \Theta^{(\alpha)}(t)$, then by convention $I_x^{(\alpha)}(t) := \emptyset$. We define a family $T := (T_x, x \in (0,1))$ by

$$T_x(t) := \inf\left\{u \geq 0 : \int_0^u |I_x^{(\alpha)}(r)|^\alpha dr > t\right\}, \quad t \geq 0.$$

For $t \geq 0$, the set $\Theta^{(\alpha)}(T(t)) := \bigcup_{x\in(0,1)} I_x^{(\alpha)}(T_x(t))$ is open since it is the union of open intervals, and the family $(\Theta^{(\alpha)}(T(t)), t \geq 0)$ is nested. So we obtain a new interval-partition fragmentation $(\Theta(t))_{t\geq 0} := (\Theta^{(\alpha)}(T(t))_{t\geq 0}$. According to Theorem 2 in [15], Θ is a homogeneous fragmentation with no erosion and the same dislocation measure ν. Let $\mathbf{X}$ be the mass fragmentation associated with Θ. We call $\mathbf{X}$ the homogeneous counterpart of $\mathbf{X}^{(\alpha)}$.

In view of future use we state the following lemma, which is an extension of Equation (6) in [16].

Lemma 2.2.1. *We consider a self-similar fragmentation $\mathbf{X}^{(\alpha)}(t) = (X_i^{(\alpha)}(t))_{i\in\mathbb{N}}$ with no erosion and index of self-similarity $\alpha \in \mathbb{R}$, and its homogeneous counterpart $\mathbf{X}(t) = (X_i(t))_{i\in\mathbb{N}}$. Let*

$f : [0, \infty) \to [0, \infty)$ *be a measurable function, then the following equality holds:*

$$\int_0^\infty \sum_{i=1}^\infty f(X_i(t))dt = \int_0^\infty \sum_{i=1}^\infty (X_i^{(\alpha)}(t))^\alpha f(X_i^{(\alpha)}(t))dt.$$

Proof. For every $t > 0$, we have

$$\sum_{i=1}^\infty f(X_i(t)) = \int_0^1 |I_x^{(\alpha)}(T_x(t))|^{-1} f(|I_x^{(\alpha)}(T_x(t))|)dx.$$

Changing variable by $s = T_x(t)$, thus $dt = |I_x^{(\alpha)}(s)|^\alpha ds$, we have

$$\int_0^\infty \sum_{i=1}^\infty f(X_i(t))dt = \int_0^1 dx \int_0^\infty |I_x^{(\alpha)}(s)|^{\alpha-1} f(|I_x^{(\alpha)}(s)|)ds = \int_0^\infty \sum_{i=1}^\infty (X_i^{(\alpha)}(s))^\alpha f(X_i^{(\alpha)}(s))ds.$$

□

Remark 2.2.2. *We consider the homogeneous fragmentation* $\mathbf{X}$ *as above. For* $p > 1$*, set*

$$\Sigma(p) := \int_0^{+\infty} \sum_{i=1}^\infty X_i(r)^p dr. \tag{2.2.1}$$

Lemma 2.2.1 implies that $\Sigma(p)$ *has the same law as*

$$\Sigma^{(1-p)}(1) := \int_0^{+\infty} \sum_{i=1}^\infty X_i^{(1-p)}(r)dr,$$

where $\mathbf{X}^{(1-p)}$ *is a self-similar fragmentation with index* $1 - p < 0$*, no erosion and the same dislocation measure* ν*. The random variable* $\Sigma^{(1-p)}(1)$ *is called the area of the fragmentation* $\mathbf{X}^{(1-p)}$*, whose law is described by Theorem 2.1 in [18]. Therefore, we also know the law of* $\Sigma(p)$*. In particular, we note that* $\Sigma(p)$ *has finite* k*-moment for* $k \in \mathbb{N}$*, see Lemma 3.1 in [18].*

2.2.3 The tagged fragment of a homogeneous fragmentation

Let $\mathbf{X} = ((X_i(t))_{i\in\mathbb{N}}, t \geq 0)$ be a homogeneous fragmentation with no erosion and dislocation measure ν. Denote the natural filtration of $\mathbf{X}$ by $(\mathcal{F}_t = \sigma(X_i(s), s \leq t))_{t\geq 0}$. In this section we recall some results about the tagged fragment taken from Section 4 of [15].

As in Section 2.2.1, let Θ be an interval fragmentation whose associated mass fragmentation $\mathbf{X}$. In particular, $\Theta_t = \bigcup_{i\in\mathbb{N}} I_i(t)$, $t \geq 0$. Given a uniform random variable V in $(0, 1)$ which is independent of Θ, the *tagged fragment* is the interval component that contains V. Denote the rank and the length of the tagged fragment at time t respectively by $n(t)$ and

$$\chi(t) := |I_{n(t)}(t)| = \sum_{i=1}^{\infty} \mathbf{1}_{\{V\in I_i(t)\}}|I_i(t)|.$$

If $V \notin \Theta(t)$, then let $n(t) = -1$ and $\chi(t) = 0$ by convention. The tagged fragment is closely related to a subordinator.

Lemma 2.2.3. *The process $\xi = -\log \chi$ is a (possibly killed) $(\mathcal{F}_t)$-subordinator with Laplace exponent*

$$\Phi(p) := \int_{\mathcal{S}} \Big(1 - \sum_{i=1}^{\infty} s_i^{p+1}\Big)\nu(d\mathbf{s}), \quad p \geq 0, \tag{2.2.2}$$

and its expected value is

$$\mathbb{E}\left[\xi(1)\right] = m := \begin{cases} \int_{\mathcal{S}} \sum_{i=1}^{\infty} s_i \log(s_i^{-1})\nu(d\mathbf{s}) & \text{when } \nu \text{ is conservative,} \\ +\infty & \text{when } \nu \text{ is dissipative.} \end{cases} \tag{2.2.3}$$

Let dU be the potential measure of ξ, whose distribution function is

$$U(x) := \mathbb{E}\left[\int_0^{\infty} \mathbf{1}_{\{\xi(t)\leq x\}} dt\right], \quad x \geq 0,$$

then the Laplace transform of dU is

$$\int_0^{\infty} \mathrm{e}^{-px} dU(x) = \Phi(p)^{-1}, \quad p \geq 0. \tag{2.2.4}$$

Further, let $f : [0,\infty) \to [0,\infty)$ be a measurable function with $f(0) = 0$, then

$$\mathbb{E}\left[\int_0^{\infty} \sum_{i=1}^{\infty} f(X_i(t))dt\right] = \mathbb{E}\left[\int_0^{\infty} \chi(t)^{-1} f(\chi(t))dt\right] = \int_0^{\infty} \mathrm{e}^{x} f(\mathrm{e}^{-x})dU(x). \tag{2.2.5}$$

Proof. Theorem 3 in [14] shows that ξ is a subordinator with Laplace exponent (2.2.2), therefore (2.2.4) and (2.2.3) follow as consequences. It is clear that conditionally on $\mathcal{F}_t$, the distribution of $\chi(t)$ is given by

$$\mathbb{P}\left[\chi(t) = X_i(t) \mid \mathcal{F}_t\right] = X_i(t), \quad \forall i \in \mathbb{N}, \qquad \mathbb{P}\left[\chi(t) = 0 \mid \mathcal{F}_t\right] = 1 - \sum_{i=1}^{\infty} X_i(t),$$

which yields (2.2.5). □

We say ξ is *lattice supported* if the law of $\xi(1)$ is a discrete measure supported by an arithmetic sequence including zero. It is clear that ξ is not lattice supported whenever ξ is not a compound Poisson process.

Definition 2.2.4. *The dislocation measure ν is "non-lattice" if ξ is not lattice supported.*

2.2.4 Large dislocations in a self-similar fragmentation

We introduce "large dislocations" in this section. As in the introduction, a dislocation of a fragmentation is labeled by $(x, \mathbf{s} = (s_1, s_2, \cdots)) \in [0,1] \times \mathcal{S}$ if in this dislocation a fragment with size x splits into a sequence of masses $(x\mathbf{s})$. We recall from the definition of $\mathcal{S}$ that $(s_1, s_2, \cdots)$ in arranged in the decreasing order.

Definition 2.2.5. *Let $\psi : [0,1] \times \mathcal{S} \to [0, +\infty)$ be a measurable function with $\psi(0, \cdot) \equiv 0$. For $\epsilon > 0$, a dislocation marked by $(x, \mathbf{s}) \in (0,1] \times \mathcal{S}$ in a fragmentation process such that $\psi(x, \mathbf{s}) > \epsilon$ is called a (ψ, ϵ)-large dislocation.*

This definition is motivated by the question about large triangles in Brownian triangulation in the Introduction. We note that by this definition, the number $N'(\epsilon)$ in Theorem 2.1.1 is the number of (ψ', ϵ)-large dislocations in the fragmentation Θ_e and $N''(\epsilon)$ is the number of (ψ'', ϵ)-large dislocations in Θ_e, where ψ' and ψ'' are defined respectively by (2.1.2) and (2.1.3), if we regard Δ as a subset of $\mathcal{S}$.

We now state the main result of this section.

Theorem 2.2.6. *Consider a self-similar fragmentation $(\mathbf{X}^{(\alpha)}(t))_{t\geq 0} = (X_1^{(\alpha)}(t), X_2^{(\alpha)}(t), \cdots)_{t\geq 0}$ of index α with no erosion and dislocation measure ν. Let ψ be a function defined as in Definition 2.2.5, and denote by $N(\epsilon)$ the total number of (ψ, ϵ)-large dislocations in $\mathbf{X}^{(\alpha)}$. Suppose that ψ can be expressed in the form*

$$\psi(x, \mathbf{s}) = \varphi(\mathbf{s}) x^b, \tag{2.2.6}$$

where $\varphi : \mathcal{S} \to [0, \infty)$ is bounded and $b > 0$. Define $g : (0, \infty) \to [0, \infty]$ by

$$g(u) := \nu\left(\mathbf{s} \in \mathcal{S} : \varphi(\mathbf{s}) > u\right), \quad u > 0, \tag{2.2.7}$$

and consider respectively the following two (mutually exclusive) situations:

- *(H1) there exists $0 < a < \frac{1}{b}$, such that $g(u) = o(u^{-a})$ as $u \to 0^+$,*
- *(H2) there exists $a > \frac{1}{b}$ and $c > 0$, such that $g(u) \sim cu^{-a}$, as $u \to 0^+$.*

Note that if $\nu(\mathcal{S}) < \infty$, then (H1) is always verified.

1. *If ν is non-lattice in the sense of Definition 2.2.4 and (H1) holds, then*

$$\lim_{\epsilon \to 0} \epsilon^{\frac{1}{b}} N(\epsilon) = \frac{1}{m} \int_0^\infty g(u^b) du \quad \text{in } L^2(\mathbb{P}),$$

where m is defined as in (2.2.3), and by convention $\frac{1}{\infty} = 0$.

2. *If (H2) holds, then*

$$\lim_{\epsilon\to 0}\epsilon^a N(\epsilon) = c\int_0^{+\infty}\sum_{i=1}^{\infty} X_i^{(\alpha)}(t)^{ab+\alpha}dt \quad \text{in } L^2(\mathbb{P}).$$

Remark 2.2.7. 1. *Theorem 2.2.1 shows a phase transition when b varies. If for $a > 0$ and $c > 0$,*

$$g(u) = \nu(\mathbf{s}\in\mathcal{S} : \varphi(\mathbf{s}) > u) \sim cu^{-a}, \quad u \to 0^+,$$

then the critical value of b is $b_c = \frac{1}{a}$. In the sub-critical phase, the scaling limit is is a constant while in the super-critical phase the limit is a random variable. Notice that this phase transition is only possible when $\nu(\mathcal{S}) = \infty$.

2. *Theorem 2.2.6 does not cover the critical case, in which there exists $c > 0$ such that $g(u) \sim cu^{-\frac{1}{b}}$ as $u \to 0^+$. In the critical case, on the one hand by comparing with the sub-critical phase we see that $\lim_{\epsilon\to 0}\epsilon^{\frac{1}{b}+\delta}N(\epsilon) = 0$ in $L^2(\mathbb{P})$ for all $\delta > 0$, so $N(\epsilon) = o(\epsilon^{-(\delta+\frac{1}{b})})$; on the other hand, by comparing with the super-critical phase we have for all $\delta > 0$ that in probability*

$$\liminf_{\epsilon\to 0}\epsilon^{\frac{1}{b}}N(\epsilon) \geq c\int_0^{+\infty}\sum_{i=1}^{\infty} X_i^{(\alpha)}(t)^{\alpha+1-\delta}dt.$$

If the fragmentation is conservative, since it follows from Lemma 2.2.1 that

$$\int_0^{+\infty}\sum_{i=1}^{\infty} X_i^{(\alpha)}(t)^{\alpha+1}dt = \int_0^{+\infty} 1dt = +\infty,$$

then $N(\epsilon) \neq \mathcal{O}(\epsilon^{-\frac{1}{b}})$. However, we do not have a finer result.

3. *The functions ψ' and ψ'' defined as in (2.1.2) and (2.1.3) do not have the form (2.2.6), thus we cannot apply directly Theorem 2.2.6 to $N'(\epsilon)$ and $N''(\epsilon)$. We will show how to overcome this difficulty in Section 2.3.*

Before tackling the proof of Theorem 2.2.6, let us look at an example of its application. It concerns *fragmentation trees* [52]. We consider a self-similar fragmentation $\mathbf{X}$ with non-lattice dislocation measure ν satisfying $\nu(\mathcal{S}) < \infty$, such that the fragmentation $\mathbf{X}$ has a discrete genealogical structure. Let us denote the *genealogical tree* by $\mathcal{U} := \bigcup_{n=0}^{\infty}\mathbb{N}^n$, where $\mathbb{N}^0 = \{\emptyset\}$ by convention. Each $u \in \mathcal{U}$ is called an *individual*, we assign to each individual a fragment in the following way. The root $\emptyset$ represents the initial state and is marked by its mass $m_\emptyset = 1$. Suppose that an individual $u \in \mathcal{U}$ stands for a fragment of mass $m_u > 0$. Since $\nu(\mathcal{S}) < \infty$, this fragment lives for a strictly positive time before it splits. When it splits, it generates fragments of masses $(m_{(u,j)})_{j\in\mathbb{N}}$. Thus for $j \in \mathbb{N}$, the j-th child of u is $(u, j) \in \mathcal{U}$ is the fragment of mass $m_{(u,j)}$, possibly zero. For $\epsilon > 0$, let the *fragmentation tree* be the sub-tree of $\mathcal{U}$ consisting all nodes with mass greater ϵ. Then the number of vertices in a fragmentation tree is the same as

$N(\epsilon)$, the number of (ψ,ϵ)-large dislocations with $\psi : [0,1]\times\mathcal{S} \to [0,\infty)$ defined by $\psi(x,\mathbf{s}) = x$. Then $g(u) = \nu(\mathcal{S})\mathbf{1}_{\{u<1\}}$ for $u > 0$. Thus (H1) holds and it follows from Theorem 2.2.6 that

$$\lim_{\epsilon\to 0} \epsilon N(\epsilon) = \frac{1}{m}\nu(\mathcal{S}) \quad \text{in } L^2(\mathbb{P}).$$

We refer to Theorem 1.3 in [52] for sharper results.

The rest of this section is devoted to the proof of Theorem 2.2.6. We first point out that it suffices to consider homogeneous fragmentations. If Theorem 2.2.6 holds for the homogeneous fragmentations, noticing that the index changing transformation defined in Section 2.2.2 preserves the number of (ψ,ϵ)-large dislocations, then we can easily obtain the desired results for self-similar fragmentations with any index $\alpha \in \mathbb{R}$ by using this transformation and Lemma 2.2.1. The details are left to the readers.

Hence we will focus on homogeneous fragmentations. We will show by Corollary 2.2.10 below that it is equivalent to study $A(\epsilon)$ defined as in (2.2.9) below. Then we will study the asymptotic behavior of $A(\epsilon)$ as $\epsilon \to 0$ respectively when (H1) holds or (H2) holds, which finally proves Theorem 2.2.6.

The compensated martingale Throughout the rest of Section 2.2, we consider a homogeneous fragmentation $\mathbf{X} = (X_i)_{i\in\mathbb{N}}$ with no erosion and dislocation measure ν, and write $(\mathcal{F}_t)_{t\geq 0}$ for its natural filtration.

A homogeneous fragmentation possesses a Poissonian structure which is described as follows. At every time $t > 0$, there is at most one fragment that splits, we denote its index by $\kappa(t)$, and denote $\mathbf{s}(t)$ for the ratio between the masses of the "children" generated in this dislocation and their "parent". Then a dislocation is characterized by a triple $(t, \kappa(t), \mathbf{s}(t)) \in [0,\infty)\times\mathbb{N}\times\mathcal{S}$. According to Theorem 9 in [11], the dislocations of $\mathbf{X}$ correspond to the atoms of a $(\mathcal{F}_t)$-Poisson point process $(\kappa(t),\mathbf{s}(t))_{t\geq 0}$ in $\mathbb{N}\times\mathcal{S}$, with characteristic measure $\#\otimes\nu$, where $\#$ denotes the counting measure of $\mathbb{N}$. Using these notations, we express the number of (ψ,ϵ)-large dislocations before time $t > 0$

$$N_t(\epsilon) = \sum_{r\in(0,t]} \mathbf{1}_{\left\{\psi\left(X_{\kappa(r)}(r-),\mathbf{s}(r)\right)>\epsilon\right\}},$$

and the number of all (ψ,ϵ)-large dislocations is $N(\epsilon) = \lim_{t\to\infty} N_t(\epsilon)$.

The Poissonian structure of the homogeneous fragmentation $\mathbf{X}$ permits us to introduce the compensated martingale. For $\epsilon > 0$, define a function $f_\epsilon : [0,\infty) \to [0,\infty]$ by

$$f_\epsilon(x) := \nu\left(\mathbf{s}\in\mathcal{S} : \psi(x,\mathbf{s}) > \epsilon\right), \quad x \in [0,\infty). \tag{2.2.8}$$

Recall that $\psi(0,\cdot) \equiv 0$, thus $f_\epsilon(0) = 0$. Set

$$A(\epsilon) := \int_0^\infty \sum_{i=1}^\infty f_\epsilon(X_i(r))dr. \tag{2.2.9}$$

If $\mathbb{E}[A(\epsilon)] < \infty$, then it follows immediately from the compensation formula (see e.g. Section O.5 in [12]) for the Poisson point process $(\kappa(t), \mathbf{s}(t))_{t\geq 0}$, that

$$M_t(\epsilon) := N_t(\epsilon) - \int_0^t \sum_{i=1}^{\infty} f_\epsilon(X_i(r))dr, \quad t \geq 0,$$

is a uniformly integrable $(\mathcal{F}_t)_{t\geq 0}$-martingale. Further,

$$M_t(\epsilon) \underset{t\to\infty}{\longrightarrow} N(\epsilon) - A(\epsilon) \quad \text{a.s. and in } L^1(\mathbb{P}).$$

In particular, we have

Lemma 2.2.8. *If* $\mathbb{E}[A(\epsilon)] < \infty$, *then* $\mathbb{E}[N(\epsilon)] = \mathbb{E}[A(\epsilon)] < \infty$.

Further, by looking at the quadratic variation of the martingale $(M_t(\epsilon))_{t\geq 0}$, we have the following lemma.

Lemma 2.2.9. *If* $\mathbb{E}[A(\epsilon)] < \infty$, *then*

$$\mathbb{E}\left[(N(\epsilon) - A(\epsilon))^2\right] = \mathbb{E}[A(\epsilon)] < \infty.$$

Proof. Since ϵ is fixed, we do not indicate the dependence on ϵ for simplicity. If $\mathbb{E}[A] < \infty$, then by Lemma 2.2.8, $\mathbb{E}[N] = \mathbb{E}[A] < \infty$. Noticing that N_t and $\int_0^t \sum_{i=1}^{\infty} f_\epsilon(X_i(r))dr$ are both increasing with respect to t, we have

$$\mathbb{E}\left[\int_0^{\infty} |dM_t|\right] \leq \mathbb{E}[N] + \mathbb{E}[A] < \infty,$$

i.e. the martingale $(M_t)_{t\geq 0}$ is of integrable variation.

We first suppose that the martingale $(M_t)_{t\geq 0}$ is bounded in $L^2(\mathbb{P})$. According to Lemma 36.2 in Chapter VI [72], since the martingale $(M_t)_{t\geq 0}$ is bounded in $L^2(\mathbb{P})$ and has integrable variation, its quadratic variation process is

$$[M]_t = \sum_{r\in(0,t]} (M_r - M_{r-})^2 = \sum_{r\in(0,t]} \mathbf{1}^2_{\left\{\psi\left(X_{\kappa(r)}(r-),\mathbf{s}(r)\right)>\epsilon\right\}} = N_t, \quad t \geq 0,$$

and $(M_t^2 - [M]_t)_{t\geq 0}$ is a uniformly integrable martingale. Thus

$$\mathbb{E}\left[(N-A)^2\right] = \lim_{t\to\infty} \mathbb{E}\left[M_t^2\right] = \lim_{t\to\infty} \mathbb{E}\left[[M]_t\right] = \mathbb{E}[N] = \mathbb{E}[A],$$

where the last equality follows from Lemma 2.2.8.

It thus remains to prove that $(M_t)_{t\geq 0}$ is indeed bounded in $L^2(\mathbb{P})$. Let us consider a sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ with $T_n := \inf\{t \geq 0 : |M_t| > n\}$ (by convention $\inf\emptyset = +\infty$). For every fixed $n \in \mathbb{N}$, the martingale $(M_{T_n\wedge t})_{t\geq 0}$ is bounded, thus it follows from the arguments above that $\mathbb{E}\left[M^2_{T_n\wedge t}\right] = \mathbb{E}\left[N_{T_n\wedge t}\right]$ for every $t \geq 0$. Then we have by Fatou's lemma that for every $t \geq 0$

$$\mathbb{E}\left[M_t^2\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[M^2_{T_n\wedge t}\right] = \liminf_{n\to\infty} \mathbb{E}\left[N_{T_n\wedge t}\right] \leq \mathbb{E}\left[N\right] = \mathbb{E}\left[A\right].$$

So we have $\sup_{t\geq 0} \mathbb{E}\left[M_t^2\right] \leq \mathbb{E}\left[A\right] < \infty$, which completes the proof. ☐

Corollary 2.2.10. *For $\lambda > 0$, if $\epsilon^\lambda A(\epsilon)$ converges in $L^2(\mathbb{P})$ as $\epsilon \to 0$, then $\epsilon^\lambda N(\epsilon)$ converges in $L^2(\mathbb{P})$ to the same limit as $\epsilon \to 0$.*

Proof. Set

$$A_0 := \lim_{\epsilon\to 0} \epsilon^\lambda A(\epsilon) \quad \text{in } L^2(\mathbb{P}).$$

By Lemma 2.2.9, we have

$$\begin{aligned}\mathbb{E}\left[(\epsilon^\lambda N(\epsilon) - A_0)^2\right] &\leq 2\mathbb{E}\left[(\epsilon^\lambda N(\epsilon) - \epsilon^\lambda A(\epsilon))^2\right] + 2\mathbb{E}\left[(\epsilon^\lambda A(\epsilon) - A_0)^2\right] \\ &= 2\epsilon^{2\lambda}\mathbb{E}\left[A(\epsilon)\right] + 2\mathbb{E}\left[(\epsilon^\lambda A(\epsilon) - A_0)^2\right].\end{aligned}$$

Then the claim holds since

$$\lim_{\epsilon\to 0} 2\epsilon^{2\lambda}\mathbb{E}\left[A(\epsilon)\right] + 2\mathbb{E}\left[(\epsilon^\lambda A(\epsilon) - A_0)^2\right] = 0.$$

☐

By Corollary 2.2.10, to study the asymptotic behavior of $N(\epsilon)$ as $\epsilon \to 0$, it suffices to study the asymptotic behavior of $A(\epsilon)$ as $\epsilon \to 0$.

The case when (H1) holds Now we study the asymptotic behavior of $A(\epsilon)$ as $\epsilon \to 0$. Suppose that ψ has the form (2.2.6), then by the definitions of f_ϵ and g in (2.2.8) and (2.2.7), we can rewrite $A(\epsilon)$ by

$$A(\epsilon) = \int_0^\infty \sum_{i=1}^\infty f_1(\epsilon^{-\frac{1}{b}} X_i(t))dt, \tag{2.2.10}$$

and we have

$$f_1(x) = g(x^{-b}), \quad x > 0. \tag{2.2.11}$$

We first suppose that (H1) holds. Let us explain briefly the motivation of considering (H1). By (2.2.5), we have

$$\mathbb{E}\left[A(\epsilon)\right] = \epsilon^{-\frac{1}{b}} \int_0^\infty f_1\left(\mathrm{e}^{(-\frac{1}{b}\log\epsilon - x)}\right) \mathrm{e}^{-(-\frac{1}{b}\log\epsilon - x)} dU(x). \tag{2.2.12}$$

Recall that dU is the potential measure of the subordinator ξ as in Lemma 2.2.3, we can study the limit of the right hand side when $\epsilon \to 0$ with the help of the renewal theorem for subordinators (see for example Proposition 1.6 in [13]). To use the renewal theorem, we need the function defined by $f_1(\mathrm{e}^x)\mathrm{e}^{-x} = g(e^{-bx})e^x$ to be directly Riemann integrable on $\mathbb{R}$. Hence it is natural to consider condition (H1), which ensures this integrability. In order to use the renewal theorem, we also suppose that ν is non-lattice in the sense of Definition 2.2.4.

Lemma 2.2.11. *Suppose that ν is non-lattice. If ψ has the form (2.2.6) and (H1) holds, then*

$$\lim_{\epsilon\to 0} \epsilon^{\frac{1}{b}} \mathbb{E}\left[A(\epsilon)\right] = \frac{1}{m}\int_0^\infty g(u^b)du,$$

where m is defined as in (2.2.3), and by convention $\frac{1}{\infty} = 0$.

Proof. If ψ has the form (2.2.6) and (H1) holds, then there exists $\bar{c} > 0$ such that

$$f_1(x) = g(x^{-b}) \le \bar{c}\mathbf{1}_{\left\{x\ge|\varphi|^{-\frac{1}{b}}\right\}} x^{ab}, \quad x > 0, \tag{2.2.13}$$

where $|\varphi|$ stands for the L^∞ norm of φ. It follows that $\tilde{f}\colon [0,\infty) \to [0,\infty)$ defined by

$$\tilde{f}(y) := f_1(\mathrm{e}^y|\varphi|^{-\frac{1}{b}})\mathrm{e}^{-y}|\varphi|^{\frac{1}{b}} \le \bar{c}|\varphi|^{\frac{1}{b}-a}\mathrm{e}^{(ab-1)y}, \quad y\in[0,\infty),$$

is directly Riemann integrable on $[0,\infty)$. Observing that $f_1(x) = 0$ for all $x < |\varphi|^{-\frac{1}{b}}$, we write (2.2.12) in terms of $\tilde{f}$ and obtain that

$$\mathbb{E}\left[A(\epsilon)\right] = \epsilon^{-\frac{1}{b}} \int_0^{\frac{1}{b}\log|\varphi| - \frac{1}{b}\log\epsilon} \tilde{f}\left(\frac{1}{b}\log|\varphi| - \frac{1}{b}\log\epsilon - y\right) dU(y).$$

As $-\frac{1}{b}\log\epsilon \to +\infty$ when $\epsilon \to 0$, by the renewal theorem (Proposition 1.6 in [13]) [1] for subordinator ξ, we have

$$\lim_{\epsilon\to 0}\epsilon^{\frac{1}{b}}\mathbb{E}\left[A(\epsilon)\right] = \frac{1}{\mathbb{E}\left[\xi(1)\right]}\int_0^{+\infty}\tilde{f}(y)dy = \frac{1}{\mathbb{E}\left[\xi(1)\right]}\int_{-\infty}^{+\infty}\mathrm{e}^{-x}f_1(\mathrm{e}^x)dx = \frac{1}{m}\int_0^{+\infty} g(u^b)du,$$

where we have changed variables $x = y - \frac{1}{b}\log|\varphi|$ and $u = \mathrm{e}^{-x}$ to get the second equality and the third equality respectively. $\square$

Further we can prove the following result.

Lemma 2.2.12. *Suppose that ν is non-lattice. If ψ has the form (2.2.6) and (H1) holds, then we have*

$$\lim_{\epsilon\to 0}\epsilon^{\frac{2}{b}}\mathbb{E}\left[A(\epsilon)^2\right] = \left(\frac{1}{m}\int_0^\infty g(u^b)du\right)^2.$$

[1] More precisely, we use the integral version of the renewal theorem, also known as the key renewal theorem, which can be derived from Proposition 1.6 in [13] by using the same arguments as the proof of Theorem 4.4.5 in [42].

We postpone the proof of Lemma 2.2.12 to Section 2.6. Our arguments proceed in a similar way as in the proof of Lemma 5 in [24].

Now we are able to prove Theorem 2.2.6 for the case when (H1) holds.

Proof of Theorem 2.2.6: when (H1) holds. By Lemma 2.2.11 and Lemma 2.2.12, we have

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\left(\epsilon^{\frac{1}{b}}A(\epsilon)-\frac{1}{m}\int_0^\infty g(u^b)du\right)^2\right]=0$$

then the claim follows from Corollary 2.2.10. □

In the same spirit as Lemma 2.2.12, we introduce the following lemma for future use in Section 2.3. We postpone its proof to Section 2.6.

Lemma 2.2.13. *Consider the (ψ,ϵ)-large dislocations of fragments of masses greater than $\frac{1}{2}$, and denote their number by $\bar{N}(\epsilon)$. If ψ has the form (2.2.6) and (H1) holds, then*

$$\lim_{\epsilon\to 0}\epsilon^{\frac{1}{b}}\bar{N}(\epsilon)=0 \quad \textit{in } L^2(\mathbb{P}).$$

The case when (H2) holds We still suppose that ψ has the form (2.2.6), which implies that (2.2.10) and (2.2.11) hold. Now we turn to the situation when (H2) holds. This situation differs significantly from the case when (H1) holds: now the function $f_1(e^x)e^{-x}$ is not directly Riemann integrable on $\mathbb{R}$, thus we cannot obtain the result in Lemma 2.2.11. However, (H2) implies that $f_1(\epsilon^{-\frac{1}{b}}x)=g(\epsilon x^{-b})\sim c\epsilon^a x^{ab}$ as $\epsilon\to 0$. As $ab>1$, recall that $\Sigma(ab)$ defined as in (2.2.1) is square integrable. Thus intuitively $A(\epsilon)\sim c\epsilon^{-a}\Sigma(ab)$ as $\epsilon\to 0$. To give a rigorous proof, let us introduce the following lemma.

Lemma 2.2.14. *Recall f_ϵ from (2.2.8). Suppose that there exist $\rho>1$, $\bar{c}>0$ and $\lambda>0$, such that for every $x\in[0,1]$,*

$$\epsilon^\lambda f_\epsilon(x)\le \bar{c}x^\rho, \quad \textit{for all } \epsilon>0, \tag{2.2.14}$$

and that for every $x\in[0,1]$, $f_*(x):=\lim_{\epsilon\to 0}\epsilon^\lambda f_\epsilon(x)$ exists. Then we have

$$\lim_{\epsilon\to 0}\epsilon^\lambda A(\epsilon)=\int_0^{+\infty}\sum_{i=1}^\infty f_*\left(X_i(t)\right)dt \quad \textit{in } L^2(\mathbb{P}).$$

We note that this lemma does not require ψ to have the form (2.2.6). We also remark that although λ is not unique, the only interesting choice of λ is the one such that $f_*\not\equiv 0$.

Proof. For $t\ge 0$, it follows from (2.2.14) that

$$\epsilon^\lambda\sum_{i=1}^\infty f_\epsilon(X_i(t))\le \bar{c}\sum_{i=1}^\infty X_i^\rho(t).$$

Recall that $\Sigma(\rho)$ defined as in (2.2.1) is a square integrable random variable, thus $\mathbb{P}$-almost surely $\bar{c}\Sigma(\rho)$ is finite. Integrate the left-hand side with respect to t, then let $\epsilon \to 0$. Hence using the dominated convergence theorem, we get that $\mathbb{P}$-almost surely

$$\lim_{\epsilon\to 0} \epsilon^{\lambda} A(\epsilon) = \lim_{\epsilon\to 0} \epsilon^{\lambda} \int_0^{\infty} \sum_{i=1}^{\infty} f_{\epsilon}(X_i(t))dt = \int_0^{\infty} \sum_{i=1}^{\infty} f_*(X_i(t))dt \le \bar{c}\Sigma(\rho).$$

Using the dominated convergence theorem, we have

$$\lim_{\epsilon\to 0} \mathbb{E}\left[\left(\epsilon^{\lambda} A(\epsilon) - \int_0^{\infty} \sum_{i=1}^{\infty} f_*(X_i(t))dt\right)^2\right] = 0.$$

$\square$

Corollary 2.2.15. *If ψ has the form* (2.2.6) *and (H2) holds, then*

$$\lim_{\epsilon\to 0} \epsilon^{a} A(\epsilon) = c\Sigma(ab) \quad \textit{in } L^2(\mathbb{P}).$$

where $\Sigma(ab)$ is defined as in (2.2.1).

Proof. If (H2) holds, then there exists $\bar{c} > 0$ such that

$$g(u) = g(u)\mathbf{1}_{\{u\le|\varphi|\}} \le \bar{c}\mathbf{1}_{\{u\le|\varphi|\}} u^{-a}, \quad \text{for all } u > 0.$$

where $|\varphi|$ stands for the L^{∞} norm of φ. Thus, recalling f_{ϵ} from (2.2.8), for every $x \in [0,1]$ we have

$$\epsilon^{a} f_{\epsilon}(x) = \epsilon^{a} g(\epsilon x^{-b}) \le \bar{c}x^{ab}, \quad \text{for all } \epsilon > 0.$$

Further, (H2) yields that for every $x \in [0,1]$,

$$\lim_{\epsilon\to 0} \epsilon^{a} f_{\epsilon}(x) = \lim_{\epsilon\to 0} \epsilon^{a} g(\epsilon x^{-b}) = cx^{ab}.$$

Hence the claim follows from Lemma 2.2.14. $\square$

Now we complete the proof of Theorem 2.2.6.

Proof of Theorem 2.2.6: when (H2) holds. Recall that we have reduced the proof to the homogeneous case, $\alpha = 0$. Then the result follows from Corollary 2.2.15 and Corollary 2.2.10. $\square$

2.3 The Brownian triangulation

In this section, we come back to the Brownian triangulation $\mathcal{L}_{\mathbf{e}}$. We will prove Theorem 2.1.1 and find the law of the length of the longest chord in $\mathcal{L}_{\mathbf{e}}$.

Recall from the introduction that $\mathcal{L}_{\mathbf{e}}$ is encoded by a normalized Brownian excursion e, i.e. almost surely

$$\mathcal{B} = \bigcup_{s \overset{e}{\sim} t, s,t \in [0,1)} [\mathrm{e}^{i2\pi s}, \mathrm{e}^{i2\pi t}],$$

where $s \overset{e}{\sim} t$ if and only if $e(s) = e(t) = \min_{r \in [s \wedge t, s \vee t]} e(r)$. We have also introduced a fragmentation process

$$\Theta_e(t) = \{s \in (0,1) : e_s > t\}, \quad t \geq 0.$$

Let us first give a formal proof of Proposition 2.1.2, that there exists a bijection between the faces of $\mathcal{L}_{\mathbf{e}}$ and the dislocations in Θ_e.

Proof of Proposition 2.1.2. For every $s \in (0,1)$, we write $cl_e(s)$ for the equivalence class under relation $\overset{e}{\sim}$. We observe that for each $s \in (0,1)$, $cl_e(s)$ contains at most three points, since $\mathcal{B}$ is a triangulation.

Now let us prove the bijection between the triangles of $\mathcal{L}_{\mathbf{e}}$ and the dislocations in Θ_e. Suppose that a triangle of $\mathcal{B}$ has vertices $\mathrm{e}^{i2\pi s_1}$, $\mathrm{e}^{i2\pi s_2}$ and $\mathrm{e}^{i2\pi s_3}$ with $s_1 < s_2 < s_3$, then $s_1 \overset{e}{\sim} s_2 \overset{e}{\sim} s_3$ and thus $cl_e(s_1) = \{s_1, s_2, s_3\}$, because $cl_e(s_1)$ cannot contain more than three points. On the one hand, by the definition of $\overset{e}{\sim}$ we see that $e(s_1) = e(s_2) = e(s_3)$. On the other hand, for every $r \in (s_1, s_2) \cup (s_2, s_3)$ we must have $e(r) > e(s_1)$: because otherwise there is $e(r) = e(s_1)$ and thus $r \in cl_e(s_1)$, which is impossible. Therefore, at time $e(s_1)$ there is a dislocation of Θ_e, the interval (s_1, s_3) splits into (s_1, s_2) and (s_2, s_3).

Conversely, if in Θ_e there is a dislocation that an interval (s_1, s_3) splits into (s_1, s_2) and (s_2, s_3), then $s_1 \overset{e}{\sim} s_2 \overset{e}{\sim} s_3$ and thus $cl_e(s_1) = \{s_1, s_2, s_3\}$. So there is triangle of $\mathcal{B}$ with vertices $\mathrm{e}^{i2\pi s_1}$, $\mathrm{e}^{i2\pi s_2}$ and $\mathrm{e}^{i2\pi s_3}$. □

According to [15], the fragmentation process Θ_e has index of self-similarity $-\frac{1}{2}$ and no erosion. Its dislocation measure ν_e, binary and conservative, is specified by

$$\nu_e(ds_1) = \frac{2}{\sqrt{2\pi s_1^3(1-s_1)^3}} ds_1 \quad 1/2 \leq s_1 < 1.$$

Let us regard ν_e as a measure on $\mathcal{S}$, thus ν_e is supported on

$$\{\mathbf{s} = (s_1, s_2, 0, \ldots) \in \mathcal{S} \ : \ s_1 + s_2 = 1\} \subset \mathcal{S}.$$

It is clear that ν_e is non-lattice in the sense of Definition 2.2.4. By calculation we also find the quantities defined as in Lemma 2.2.3:

$$m = \int_{\mathcal{S}} \sum_{i=1}^{\infty} s_i \log(s_i^{-1}) \nu_e(d\mathbf{s}) = 2\sqrt{2\pi}, \tag{2.3.1}$$

$$\Phi(p) = \int_{\mathcal{S}} \left(1 - \sum_{i=1}^{\infty} s_i^{p+1}\right) \nu_e(d\mathbf{s}) = 2\sqrt{2}\frac{\Gamma(p+1/2)}{\Gamma(p)}, \quad p > 0, \tag{2.3.2}$$

where Γ is the Gamma function.

2.3.1 Proof of Theorem 2.1.1

In the introduction, we have marked a dislocation in Θ_e by $(x,(s_1,s_2)) \in (0,1] \times \Delta$, if in this dislocation an interval of length x splits into two pieces of respective lengths xs_1 and xs_2. To make notations consistent with Section 2.2, let us mark a dislocation in Θ_e by $(x,(s_1,s_2,0,\cdots)) \in (0,1] \times \mathcal{S}$ from now on.

Proof of Theorem 2.1.1. **1.** In our fragmentation point of view, $N'(\epsilon)$ equals the number of all (ψ',ϵ)-large dislocations of Θ_e, where ψ' as in (2.1.2) is defined by

$$\psi'(x,\mathbf{s}) = \min(2\sin(\pi x), 2\sin(\pi x s_1), 2\sin(\pi x s_2)), \quad (x,\mathbf{s}) \in [0,1] \times \mathcal{S}.$$

We cannot directly apply Theorem 2.2.6 to $N'(\epsilon)$, because ψ' is not of form (2.2.6). Let us consider $\psi_1 : [0,1] \times \mathcal{S} \to \mathbb{R}_+$, a function defined by $\psi_1(x,\mathbf{s}) = (1-s_1)x$, $(x,\mathbf{s}) \in [0,1] \times \mathcal{S}$. This function is of form (2.2.6). Hence we will study $N^{\psi_1}(\epsilon)$, the number of (ψ_1,ϵ)-large dislocations of Θ_e, and compare $N'(\epsilon)$ with $N^{\psi_1}(\epsilon)$.

Recall that $s_1 \geq s_2$ by the definition of space $\mathcal{S}$. On the one hand, if $x \leq \frac{1}{2}$, then $\sin(\pi x s_2) \leq \sin(\pi x s_1) \leq \sin(\pi x)$, thus

$$\left\{(x,\mathbf{s}) \in (0,1] \times \mathcal{S} : 2\sin(\pi x s_2) > \epsilon, x \leq \frac{1}{2}\right\} \subset \left\{(x,\mathbf{s}) \in (0,1] \times \mathcal{S} : \psi'(x,\mathbf{s}) > \epsilon\right\}.$$

On the other hand, it is plain that $\psi'(x,\mathbf{s}) \leq 2\sin(\pi x s_2)$ and thus

$$\left\{(x,\mathbf{s}) \in (0,1] \times \mathcal{S} : \psi'(x,\mathbf{s}) > \epsilon\right\} \subset \left\{(x,\mathbf{s}) \in (0,1] \times \mathcal{S} : 2\sin(\pi x s_2) > \epsilon\right\}.$$

We may assume that $\epsilon < 2$, so $\arcsin(\epsilon/2)$ is well-defined. Let $\bar{N}^{\psi_1}(\epsilon)$ be the number of the dislocations which are (ψ_1,ϵ)-large and whose marks $(x,\mathbf{s}) \in (0,1] \times \mathcal{S}$ satisfy $x > \frac{1}{2}$. Then the above observations yield

$$N^{\psi_1}(\pi^{-1}\arcsin(\epsilon/2)) - \bar{N}^{\psi_1}(\pi^{-1}\arcsin(\epsilon/2)) \leq N'(\epsilon) \leq N^{\psi_1}(\pi^{-1}\arcsin(\epsilon/2)). \tag{2.3.3}$$

Let us look at $N^{\psi_1}(\epsilon)$. Since ψ_1 is of form (2.2.6) and g_1 defined as in (2.2.7) is

$$g_1(u) := \nu_e(\mathbf{s} \in \mathcal{S} : 1 - s_1 > u) = \frac{2\sqrt{2}}{\sqrt{\pi}} u^{-\frac{1}{2}} \frac{1-2u}{\sqrt{1-u}} \mathbf{1}_{\{u<\frac{1}{2}\}} \sim \frac{2\sqrt{2}}{\sqrt{\pi}} u^{-\frac{1}{2}}, \quad u \to 0^+, \tag{2.3.4}$$

the hypothesis (H1) is satisfied. We also find that $\int_0^\infty g_1(u)du = \frac{4\sqrt{2}}{\sqrt{\pi}}\sqrt{(1-u)u}\Big|_0^{\frac{1}{2}} = \frac{2\sqrt{2}}{\sqrt{\pi}}$. Thus using Theorem 2.2.6 and (2.3.1) yields

$$\lim_{\epsilon\to 0} \epsilon N^{\psi_1}(\epsilon) = \frac{1}{\pi} \quad \text{in } L^2(\mathbb{P}).$$

Further, since (H1) holds, applying Lemma 2.2.13 to $\bar{N}_\infty^{\psi_1}(\epsilon)$, we have

$$\lim_{\epsilon\to 0} \epsilon \bar{N}^{\psi_1}(\epsilon) = 0 \quad \text{in } L^2(\mathbb{P}).$$

Combining these two limits and (2.3.3), we prove the claim.

2. Recall that $N''(\epsilon)$ is the number of all (ψ'', ϵ)-large dislocations of Θ_e, where ψ'' defined as in (2.1.3) is

$$\psi''(x, \mathbf{s}) := 2\sin(\pi x s_1)\sin(\pi x s_2)\sin(\pi x), \quad (x, \mathbf{s}) \in [0,1]\times\mathcal{S}.$$

This function is not of form (2.2.6), thus we cannot use Theorem 2.2.6 directly. However, we will study $N''(\epsilon)$ by using similar arguments as in the proof of Theorem 2.2.6.

Because the transformation explained in Section 2.2.2 does not affect the number of total large dislocations, we regard $N''(\epsilon)$ as the number of (ψ'', ϵ)-large dislocations in $\mathbf{X}$, the homogeneous mass fragmentation counterpart of Θ_e. For $(x, (s_1, s_2, 0, \cdots)) \in (0,1]\times\mathcal{S}$ such that $s_1 + s_2 = 1$, by the trigonometric addition formula $\cos(z_1 - z_2) - \cos(z_1 + z_2) = 2\sin(z_1)\sin(z_2)$, we have

$$\psi''(x, \mathbf{s}) = 2\sin(\pi x s_1)\sin(\pi x s_2)\sin(\pi x) = (\cos(\pi x - 2\pi x s_2) - \cos(\pi x))\sin(\pi x).$$

Hence for every $0 < x \le 1$, $\psi''(x, \mathbf{s}) > \epsilon$ if and only if

$$1 - s_1 = s_2 > h''(x, \epsilon) := \frac{1}{2} - \frac{1}{2\pi x}\arccos\left(\min\left(\cos\pi x + \frac{\epsilon}{\sin\pi x}, 1\right)\right).$$

Note that when $\epsilon \ge \sin(\pi x)(1 - \cos(\pi x))$, this inequality reads $s_2 > \frac{1}{2}$ when means for all $\mathbf{s}\in\mathcal{S}$ it is impossible to have $\psi''(x, \mathbf{s}) > \epsilon$. We want to use Lemma 2.2.14 to $A''(\epsilon)$ defined as in (2.2.9):

$$A''(\epsilon) = \int_0^\infty \sum_{i=1}^\infty \nu_e\left(\mathbf{s}\in\mathcal{S} : \psi''(X_i(t), \mathbf{s}) > \epsilon\right) dt.$$

So we check the two assumptions in Lemma 2.2.14. On the one hand, it is clear that for every $x > 0$,

$$\lim_{\epsilon\to 0} h''(x, \epsilon)\epsilon^{-1} = (2\pi x \sin^2 \pi x)^{-1},$$

then using the function g_1 defined in (2.3.4), we have for every $\epsilon > 0$

$$\lim_{\epsilon\to 0} \epsilon^{\frac{1}{2}} \nu_e \left(\mathbf{s} \in \mathcal{S} : \psi''(x, \mathbf{s}) > \epsilon\right) = \lim_{\epsilon\to 0} \epsilon^{\frac{1}{2}} g_1(h''(x, \epsilon)) = 4x^{\frac{1}{2}} \sin \pi x.$$

On the other hand, observing that

$$\psi''(x, \mathbf{s}) = 2\sin(\pi x s_1)\sin(\pi x s_2)\sin(\pi x) \leq 2\pi^3 x^3 s_2,$$

and $g_1(u) \leq \frac{2\sqrt{2}}{\sqrt{\pi}} u^{-\frac{1}{2}}$ for every $u > 0$, we have

$$\epsilon^{\frac{1}{2}} \nu_e \left(\mathbf{s} \in \mathcal{S} : \psi''(x, \mathbf{s}) > \epsilon\right) \leq \epsilon^{\frac{1}{2}} g_1(\epsilon(2\pi^3 x^3)^{-1}) \leq 4\pi x^{\frac{3}{2}}.$$

Hence it follows from Lemma 2.2.14 and Corollary 2.2.10 that

$$\lim_{\epsilon\to 0} \epsilon^{\frac{1}{2}} N''(\epsilon) = \int_0^\infty \sum_{i=1}^\infty 4X_i(t)^{\frac{1}{2}} \sin(\pi X_i(t)) dt \quad \text{in } L^2(\mathbb{P}).$$

By applying Lemma 2.2.1 to Θ_e (with index of self-similarity $-\frac{1}{2}$) and its homogeneous counterpart $\mathbf{X}$, we rewrite the right-hand side in terms of $\Theta_e(t) = \bigcup_{i\in\mathbb{N}} I_i(t)$ and obtain the desired result. □

2.3.2 The length of the longest chord

In [3], Aldous has determined the law of the length of the longest chord by approximating the Brownian triangulation $\mathcal{L}_\mathbf{e}$ by discrete uniform triangulations of polygons, studying uniform triangulations and then passing to the limit. Here we propose another approach using the bijection in Proposition 2.1.2.

We will make use of the *centroid*, the face that contains the origin, since it is plain that the longest chord must be an edge of the centroid. Almost surely, no chord in $\mathcal{L}_\mathbf{e}$ passes through the origin thus the centroid in $\mathcal{L}_\mathbf{e}$ is unique. Let us consider the dislocation in Θ_e associated with the centroid. By Proposition 2.1.2, if it is marked by $(x, \mathbf{s}) \in (0, 1] \times \mathcal{S}$, then the vertices of the centroid divide the unit circle into three arcs whose lengths are $(2\pi(1-x),\ 2\pi x s_1,\ 2\pi x s_2)$. Due to the property of the centroid, each of these arcs has length less than π, then

$$\begin{aligned}(1-x) < \frac{1}{2} \ \&\ x s_1 < \frac{1}{2} \ \&\ x s_2 < \frac{1}{2} \quad &\Longleftrightarrow \quad x > \frac{1}{2} \ \&\ x s_1 < \frac{1}{2} \\ &\Longleftrightarrow \quad \min(x, 1 - x s_1) > \frac{1}{2}. \end{aligned} \tag{2.3.5}$$

Conversely, it is easy to see that if a dislocation in Θ_e is marked by $(x, \mathbf{s}) \in (0, 1] \times \mathcal{S}$ verifying the above relation, then it must correspond to the centroid. By further study of the centroid we have the following observation.

Lemma 2.3.1. *Let $2\pi L$ be the length of the minor arc with the same endpoints as the longest chord in $\mathcal{L}_{\mathbf{e}}$, then $L \le \frac{1}{2}$ and the longest chord has length $2\sin(\pi L)$. Define a function $\psi_L : [0,1] \times \mathcal{S} \to [0,\infty)$ by $\psi_L(x,\mathbf{s}) = \min(x, 1 - xs_1)$. For $a \le \frac{1}{2}$, let $N(1-a)$ be the number of $(\psi_L, 1-a)$-large dislocations in Θ_e as in Definition 2.2.5, then*

$$\mathbb{P}\left(L < a\right) = \mathbb{E}\left[N(1-a)\right]$$

Proof. A dislocation is $(\psi_L, 1-a)$-large if and only if its mark $(x,\mathbf{s}) \in (0,1] \times \mathcal{S}$ satisfies

$$\begin{aligned} \min(x, 1 - xs_1) > 1 - a \quad &\iff \quad x > 1 - a \;\&\; xs_1 < a \\ &\iff \quad (1-x) < a \;\&\; xs_1 < a \;\&\; xs_2 < a. \end{aligned} \tag{2.3.6}$$

As $(1-a) \ge \frac{1}{2}$, if there is a $(\psi_L, 1-a)$-large dislocation in Θ_e, then it is associated with the centroid by (2.3.5). In particular, almost surely $N(1-a) = 1$ or 0.

Let us consider the the dislocation in Θ_e associated with the centroid in $\mathcal{L}_{\mathbf{e}}$, whose mark is $(x,\mathbf{s}) \in [0,1] \times \mathcal{S}$. The longest chord in $\mathcal{L}_{\mathbf{e}}$ must be an edge of the centroid, thus $L \in \{(1-x), xs_1, xs_2\}$. If $N(1-a) = 1$, then this dislocation is $(\psi_L, 1-a)$-large thus $(x,\mathbf{s})$ verifies (2.3.6), which implies $L < a$. If $N(1-a) = 0$, then this dislocation is not $(\psi_L, 1-a)$-large, thus $\max(1-x, xs_1, xs_2) \ge a$, which implies $L \ge a$. Hence we conclude that $\mathbb{P}\left(L < a\right) = \mathbb{E}\left[N(1-a)\right]$. $\square$

To determine the law of the length of the longest chord in $\mathcal{L}_{\mathbf{e}}$, we still need to calculate $\mathbb{E}\left[N(1-a)\right]$ explicitly. By the transformation in Section 2.2.2, which preserves the total number of large dislocations, we may regard $N(1-a)$ as the number of $(\psi_L, 1-a)$-large dislocations of the homogeneous counterpart $\mathbf{X}$ of Θ_e. By Lemma 2.2.8,

$$\mathbb{E}\left[N(1-a)\right] = \mathbb{E}\left[\int_0^\infty \sum_{i=1}^\infty g_1(1 - a/X_i(r)) \mathbf{1}_{\{X_i(r) > 1-a\}} dr\right],$$

where function $g_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is given by (2.3.4). By (2.2.5), the right-hand side equals

$$\mathbf{1}_{\{a > \frac{1}{3}\}} \frac{2\sqrt{2}}{\sqrt{\pi}} \int_{\log \frac{1}{2a}}^{\log \frac{1}{1-a}} \mathrm{e}^x \frac{2a\mathrm{e}^x - 1}{\sqrt{(1 - a\mathrm{e}^x)\mathrm{e}^x a}} dU(x). \tag{2.3.7}$$

where, according to Lemma 2.2.3 and (2.3.2), the measure dU is characterized by its Laplace transform

$$\int_0^\infty \mathrm{e}^{-px} dU(x) = \left(\int_{\mathcal{S}} \left(1 - \sum_{i=1}^\infty s_i^{p+1}\right) \nu_e(d\mathbf{s})\right)^{-1} = \frac{1}{2\sqrt{2}} \frac{\Gamma(p)}{\Gamma(p+1/2)}, \quad p > 0.$$

Noticing that

$$\int_0^{+\infty} \mathrm{e}^{-px}(1-\mathrm{e}^{-x})^{-1/2}dx = \Gamma(1/2)\frac{\Gamma(p)}{\Gamma(p+1/2)}, \quad p>0,$$

we have

$$dU(x) = \frac{1}{2\sqrt{2\pi}}(1-\mathrm{e}^{-x})^{-1/2}dx, \quad x\in[0,\infty).$$

Hence, rewriting (2.3.7), we recover formula (9) in [3]:

$$\begin{aligned}\mathbb{P}\left(L<a\right) &= \mathbf{1}_{\{a>\frac{1}{3}\}}\frac{1}{a\pi}\int_{\frac{1}{2}}^{\frac{a}{1-a}}\frac{2y-1}{\sqrt{(1-y)(y-a)}}dy \\ &= \mathbf{1}_{\{a>\frac{1}{3}\}}\left(6\pi^{-1}(\arctan(3^{-\frac{1}{2}})-\arctan((1-2a)^{\frac{1}{2}}))-\frac{(3a-1)(1-2a)^{\frac{1}{2}}}{\pi a(1-a)}\right), \quad 0<a<\frac{1}{2}.\end{aligned}$$

2.4 Random stable laminations of a disk

In this section, we generalize our work about the Brownian triangulation to the stable laminations. Specifically, we will study the number of their large faces and find the laws of the lengths of their longest chords.

A *(geodesic) lamination of the disk* $\mathbb{D}$ is a closed subset of $\mathbb{D}$, which can be written as the union of a random collection of non-crossing chords of the circle $\partial\mathbb{D}$. In particular, a triangulation is a lamination. Conversely, it is easy to see that a lamination is a triangulation if and only if it is maximal for the inclusion relation among the laminations of $\mathbb{D}$.

The *random stable lamination of the disk with parameter* $\beta\in(1,2]$ (or shortly *β-stable lamination*) is a random model of laminations which was introduced by Kortchemski [54]. For $\beta=2$, the β-stable lamination is the Brownian triangulation. Hence from now on, we consider the β-stable lamination with $\beta\in(1,2)$. It has been shown in [54] that the β-stable lamination is the distributional limit for the Hausdorff topology of certain families of random dissections of polygon P_n when $n\to\infty$, which we do not describe here. But let us briefly present two other constructions in [54] that connect the β-stable lamination with the β-stable process and the β-stable tree.

Let ξ be a strictly β-stable spectrally positive Lévy process, whose Laplace transform is

$$\mathbb{E}\left[\exp(-p\xi_t)\right] = \exp(p^\beta t), \quad \text{for } t,p\geq 0.$$

Let ξ^{exc} be the normalized excursion of the Lévy process ξ (see Chapter VIII in [12] and Section 2.1 of [54]). For $0\leq s<t\leq 1$, we define

$$s\overset{\xi^{exc}}{\simeq} t, \text{ if } t=\inf\{u>s;\xi^{exc}_u\leq\xi^{exc}_{s-}\}.$$

Then $L_{X^{exc}} := \bigcup_{s\overset{\xi^{exc}}{\simeq} t}[\mathrm{e}^{2\pi is},\mathrm{e}^{2\pi it}]$ is the β-stable lamination.

The random stable lamination can also be encoded by a random continuous function $(H_t^{exc})_{t\in[0,1]}$. A way to define H^{exc} is as follows. For every $t \geq 0$, consider the time-reversed process

$$\xi^{(t)}(s) := \xi(t) - \xi((t-s)-), \quad 0 \leq s < t.$$

Let $S^{(t)}(s) := \sup_{0\leq u\leq s} \xi^{(t)}(s)$, the we define *the height process* H_t to be the local time at level 0 at time t of the process $S^{(t)} - \xi^{(t)}$. It is known from Lemma 1.1.3 in [41] that for every $t \geq 0$ there is the following approximation of H_t:

$$H_t = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}_{\left\{\xi_s \leq \inf_{u\in[s,t]} \xi_u + \epsilon\right\}} ds,$$

where the limit holds in probability. Almost surely the process H has a continuous modification (Theorem 1.4.3 in [41]), which will be implicitly considered. Though H is in general not a Markov process (except the Brownian case), Section 3.2 in [40] shows that, similar to Itô's excursion theory for Markovian processes, one can defined *the normalized excursion* $H^{exc} = (H_t^{exc})_{t\in[0,1]}$ of H. Recall that ξ^{exc} is the normalized excursion of ξ, then H^{exc} is given by

$$H_t^{exc} := \lim_{\epsilon\to 0} \frac{1}{C(\epsilon)} Card \left\{ u \in [0,t] :\ \xi_{u-}^{exc} < \inf_{[u,t]} \xi^{exc}; \Delta\xi_u^{exc} > \epsilon \right\}, \quad a.s.$$

where $C(\epsilon) := \frac{\beta}{\Gamma(2-\beta)\epsilon^{\beta-1}}$. We stress that H^{exc} is a continuous function. More details about H^{exc} can be found in Section 3.2 in [40] and Section 4 in [54].

Next, for $s,t \in [0,1]$, we define a relation $s \overset{H^{exc}}{\approx} t$ as follows. We first define $s \overset{H^{exc}}{\sim} t$ if and only if $H^{exc}(s) = H^{exc}(t) = \min_{r\in[s\wedge t, s\vee t]} H^{exc}(r)$ (which is the same as the relation to encode the Brownian triangulation), then we say $s \overset{H^{exc}}{\approx} t$, if $s \overset{H^{exc}}{\sim} t$ and one of the two following properties is satisfied:

1. $\forall r \in (s\wedge t, s\vee t)$, $H^{exc}(r) > H^{exc}(s) = H^{exc}(t)$,
2. Let $cl_{H^{exc}}(s) := \{r | r \overset{H^{exc}}{\sim} s\}$, then $cl_{H^{exc}}(s) \subset [s\wedge t, s\vee t]$.

According to Theorem 4.5 in [54], almost surely

$$L_{H^{exc}} := \bigcup_{s \overset{H^{exc}}{\approx} t} [\mathrm{e}^{2\pi i s}, \mathrm{e}^{2\pi i t}] = L_{\xi^{exc}}.$$

Hence the β-stable lamination can be represented by either $L_{\xi^{exc}}$ or $L_{H^{exc}}$, and we will not distinguish them.

Further, these representations imply the connection between the β-stable lamination and the fragmentation process

$$\Theta_\beta(t) = \{s \in (0,1) : H^{exc}(s) > t\}, \quad t \geq 0.$$

According to Proposition 1 and Theorem 1 in [64], this process is a self-similar fragmentation with index $(1/\beta - 1) < 0$, no erosion and dislocation measure

$$\nu_\beta(d\mathbf{s}) = D_\beta \mathbb{E}\left[T_1; \frac{\Delta T_{[0,1]}}{T_1} \in d\mathbf{s}\right], \tag{2.4.1}$$

where $(T_x)_{x\geq 0}$ is a β^{-1}-stable subordinator, $\Delta T_{[0,1]} = (\Delta_1, \Delta_2, \cdots)$ is the vector of jumps of T before time 1 reordered in the decreasing order, and $D_\beta = \frac{\beta^2\Gamma(2-1/\beta)}{\Gamma(2-\beta)}$. The connection between the β-stable lamination and Θ_β is described by the following statement.

Proposition 2.4.1. *Almost surely, there is a bijection between the faces in the β-stable lamination of $\mathbb{D}$ and the dislocations of the fragmentation Θ_β. If a face corresponds to a dislocation labeled by $(x, \mathbf{s}) \in [0,1]\times\mathcal{S}$, then its vertices divide $\partial\mathbb{D}$ into arcs of lengths $2\pi(1-x, xs_1, xs_2, \cdots)$, and the edges of this face have lengths $(2\sin(\pi x), 2\sin(\pi x s_1), 2\sin(\pi x s_2), \cdots)$.*

Proof. According to Proposition 3.10 in [54], there is a bijection between the faces of $L_{X^{exc}}$ and the jump time of X^{exc}. It is clear that the faces of $L_{H^{exc}}$ correspond to a subset of

$$\{cl_{H^{exc}}(u), u \in [0,1] : Card(cl_{H^{exc}}(u)) \geq 3\}.$$

Finally, by Proposition 4.4 in [54], the latter set corresponds to a subset of the jump time of X^{exc}. Hence we have the bijection. The second assertion is plain by the geometry. □

2.4.1 Large faces in the stable laminations

Thanks to Proposition 2.4.1, we can study the number of large faces in the β-stable lamination, $\beta \in (1,2)$. We would like to find a result of type Theorem 2.1.1. However, almost surely every face in the β-stable lamination has infinitely many sides, hence the shortest edge of a face is meaningless. On the other side, we find that the face corresponding to a dislocation labeled by $(x, \mathbf{s}) \in [0,1]\times\mathcal{S}$ has area $\frac{1}{2}\sin(2\pi x) + \frac{1}{2}\sum_{i\in\mathbb{N}}\sin(2\pi x s_i)$. If we want to estimate the number of faces of large area, then we have to study for every $x \in [0,1]$, the asymptotic behavior as $\epsilon \to 0$ of

$$f_\epsilon(x) = \nu_\beta\Big(\mathbf{s} \in \mathcal{S} \ : \ \frac{1}{2}\sin(2\pi x) + \frac{1}{2}\sum_{i\in\mathbb{N}}\sin(2\pi x s_i) > \epsilon\Big),$$

which seems rather difficult. Therefore, let us define alternatively the large faces in the following way.

For each face, we consider the corresponding **minor** arcs of its edges. For $\epsilon > 0$, we define a face to be ϵ-*large* if at least two of those arcs are longer than ϵ, and the total length of the rest arcs is greater than ϵ. This definition should be meaningful. In the Brownian triangulation case ($\beta = 2$), the triangles with the shortest edges longer than ϵ are exactly the so-defined ϵ'-large faces with $\epsilon' = 2\sin(\pi\epsilon)$. We have the following result for the number of ϵ-large faces in the β-stable lamination.

Theorem 2.4.2. *For $\beta \in (1,2)$, let $N(\epsilon)$ be the number of ϵ-large faces defined as above in the β-stable lamination, then*

$$\lim_{\epsilon\to 0} \epsilon N(\epsilon) = \frac{2\pi(\beta-1)}{\Gamma(2-\beta)} \mathbb{E}\left[\min(T_1-\Delta_1, \Delta_1)\right] \quad in\ L^2(\mathbb{P}),$$

where T_1 is the value of the β^{-1}-stable subordinator T at time 1, and Δ_1 is the largest jump of T before time 1.

Before tackling the proof, let us look at the value of $\mathbb{E}\left[\min(T_1-\Delta_1,\Delta_1)\right]$. We refer to [69] for the joint law of (T_1,Δ_1), in which it has been proved that the joint law of (T_1,Δ_1) has a density, although this density function is not explicitly given. Let us calculate $\mathbb{E}\left[\min(T_1-\Delta_1,\Delta_1)\right]$ by using the approach in [69]. By the Lévy-Itô decomposition, we find that T_1 is the sum of the atoms of a Poisson random measure on $(0,+\infty)$ with intensity $C_\beta dr/r^{1+1/\beta}$, where $C_\beta = (\beta\Gamma(1-\beta^{-1}))^{-1}$. Thus for $y>0$, the probability that no atom has mass greater than y is

$$\mathbb{P}\left(\Delta_1 \le y\right) = \exp\Big(-\int_y^\infty C_\beta dr/r^{1+1/\beta}\Big) = \exp\Big(-\frac{1}{\Gamma(1-1/\beta)} y^{-1/\beta}\Big).$$

By the restriction property of Poisson point processes, we see that the conditional distribution of $T_1-\Delta_1$ given $\Delta_1 = y$ is a subordinator with Lévy measure $C_\beta \mathbf{1}_{\{r\le y\}} dr/r^{1+1/\beta}$. Write $\tilde{T}_1^y$ for this subordinator, which is characterized by its Laplace exponent

$$\mathbb{E}\left[\exp(-p\tilde{T}_1^y)\right] = \exp\Big(-\int_0^y C_\beta/r^{1+1/\beta}(1-e^{-pr})dr\Big), \quad p\ge 0,$$

then we have

$$\mathbb{E}\left[\min(T_1-\Delta_1,\Delta_1)\right] = \int_0^\infty \mathbb{E}\left[\min(\tilde{T}_1^y, y)\right] \exp\Big(-\frac{1}{\Gamma(1-1/\beta)} y^{-1/\beta}\Big) C_\beta y^{-(1+1/\beta)} dy.$$

Proof of Theorem 2.4.2. Define $\psi_* : [0,1]\times\mathcal{S} \to [0,\infty)$ by $\psi_*(x,\mathbf{s}) = \min(s_1, 1-s_1)x$ and let $N^{\psi_*}(\epsilon)$ be the number of all (ψ_*,ϵ)-large dislocations. Let us compare $N^{\psi_*}(\frac{\epsilon}{2\pi})$ with $N(\epsilon)$. For a face associated with a dislocation marked by $(x,\mathbf{s}) \in [0,1]\times\mathcal{S}$, the corresponding shorter arcs of the edges have lengths

$$\min(2\pi(1-x), 2\pi x),\ \ \min(2\pi(1-xs_1), 2\pi x s_1),\ \ 2\pi x s_2 = \min(2\pi(1-xs_2), 2\pi x s_2),\ \ \cdots$$

When $x<\frac{1}{2}$, the two longest arcs have lengths $2\pi x$ and $2\pi x s_1$. Thus if a dislocation is $(\psi_*, \frac{\epsilon}{2\pi})$-large and its mark $(x,\mathbf{s}) \in [0,1]\times\mathcal{S}$ verifies $x<\frac{1}{2}$, then its corresponding face is ϵ-large. As before, let $\bar{N}^{\psi_*}(\epsilon)$ be the number of (ψ_*,ϵ)-large dislocations of fragments of masses greater than $\frac{1}{2}$. Then

$$N^{\psi_*}(\frac{\epsilon}{2\pi}) - \bar{N}^{\psi_*}(\frac{\epsilon}{2\pi}) \le N(\epsilon). \tag{2.4.2}$$

On the other hand, if a dislocation with mark $(x, \mathbf{s}) \in [0,1] \times \mathcal{S}$ is not $(\psi_*, \frac{\epsilon}{2\pi})$-large, then $xs_1 < \frac{\epsilon}{2\pi}$ or $x(1-s_1) < \frac{\epsilon}{2\pi}$. If $xs_1 < \frac{\epsilon}{2\pi}$, then, as $2\pi xs_i \leq 2\pi xs_1 < \epsilon$ for $i \geq 2$, at most one arc is longer than ϵ, thus the face is not ϵ-large; if $x(1-s_1) < \frac{\epsilon}{2\pi}$, noticing that $2\pi x(1-s_1)$ is the total length of all the arcs except the two with lengths $\min(2\pi xs_1, 2\pi - 2\pi xs_1)$ and $\min(2\pi x, 2\pi - 2\pi x)$, we find the face not ϵ-large. Hence

$$N(\epsilon) \leq N^{\psi_*}(\frac{\epsilon}{2\pi}). \tag{2.4.3}$$

Next we study $N^{\psi_*}(\epsilon)$ and $\bar{N}^{\psi_*}(\epsilon)$. By Section 4.4 in [49], $\exists C > 0$ such that

$$\nu_\beta(\mathbf{s} \in \mathcal{S} : \min(s_1, 1-s_1) > u) \leq \nu_\beta(\mathbf{s} \in \mathcal{S} : 1 - s_1 > u) \sim Cu^{-(1-1/\beta)}, \quad u \to 0^+.$$

Since ψ_* has the form (2.2.6) and $(H1)$ holds as $1 - 1/\beta < 1$, by Theorem 2.2.6 we have

$$\lim_{\epsilon \to 0} \epsilon N^{\psi_*}(\frac{\epsilon}{2\pi}) = 2\pi \frac{1}{m} \int_0^1 \nu_\beta\Big(\mathbf{s} \in \mathcal{S} : \min(s_1, 1-s_1) > u\Big) du \quad \text{in } L^2(\mathbb{P}), \tag{2.4.4}$$

where $m = \int_{\mathcal{S}} \sum_{i=1}^{\infty} s_i \log(s_i^{-1}) \nu_\beta(ds)$. Using Lemma 2.2.13, we get

$$\lim_{\epsilon \to 0} \epsilon \bar{N}^{\psi_*}(\epsilon) = 0 \quad \text{in } L^2(\mathbb{P}). \tag{2.4.5}$$

Then, combining (2.4.2), (2.4.3), (2.4.4) and (2.4.5), we have

$$\lim_{\epsilon \to 0} \epsilon N(\epsilon) = 2\pi \frac{1}{m} \int_0^1 \nu_\beta\Big(\mathbf{s} \in \mathcal{S} : \min(s_1, 1-s_1) > u\Big) du \quad \text{in } L^2(\mathbb{P}).$$

To compute the value of limit, let us introduce the size biased picked jump Δ^* among $(\Delta_i)_{i \in \mathbb{N}}$, the jumps of a β^{-1}-stable subordinator T before time 1. The law of Δ^* conditionally on $(\Delta_i)_{i \in \mathbb{N}}$ is given by

$$\mathbb{P}\left[\Delta^* = \Delta_j \mid (\Delta_i)_{i \in \mathbb{N}}\right] = \frac{\Delta_j}{T_1}, \quad \forall j \in \mathbb{N}.$$

Then, on the one hand, we deduce from (2.4.1) that

$$m = \int_{\mathcal{S}} \sum_{i=1}^{\infty} s_i \log(s_i^{-1}) \nu_\beta(d\mathbf{s}) = D_\beta \mathbb{E}\left[T_1 \sum_{i=1}^{\infty} \frac{\Delta_i}{T_1} \log(\frac{T_1}{\Delta_i})\right] = \mathbb{E}\left[T_1 \log(\frac{T_1}{\Delta^*})\right].$$

According to Lemma 1 in [64], the joint law of (Δ^*, T_1) has density:

$$\frac{C_\beta q_1(s-y)}{sy^{1/\beta}} dyds, \quad (y, s) \in [0, \infty)^2,$$

where the constant $C_\beta = \frac{1}{\beta\Gamma(1-1/\beta)}$, and q_1 is the density function of T_1. Then

$$\begin{aligned}
m &= D_\beta C_\beta \int_{[0,\infty)^2} \log(s/y) q_1(s-y) y^{-\frac{1}{\beta}} dy ds \\
&= D_\beta C_\beta \int_0^\infty q_1(u) du \int_u^\infty \log(\frac{s}{s-u})(s-u)^{-\frac{1}{\beta}} ds \\
&= D_\beta C_\beta \frac{\beta}{\beta-1} \frac{\pi}{\sin(\pi/\beta)} \mathbb{E}\left[T_1^{1-1/\beta}\right].
\end{aligned}$$

It is well-known that $\mathbb{E}\left[T_1^{1-1/\beta}\right] = \frac{\Gamma(2-\beta)}{\Gamma(1/\beta)}$ and $\Gamma(1-1/\beta)\Gamma(1/\beta) = \frac{\pi}{\sin(\pi/\beta)}$, so we find that

$$m = D_\beta \frac{\Gamma(2-\beta)}{\beta-1}.$$

On the other hand, by (2.4.1) and Fubini's Theorem, we have

$$\begin{aligned}
\int_0^1 \nu_\beta\Big(\mathbf{s} \in \mathcal{S} : \min(s_1, 1-s_1) > u\Big) du &= D_\beta \mathbb{E}\left[T_1 \int_0^1 \mathbf{1}_{\left\{\min(\frac{T_1-\Delta_1}{T_1}, \frac{\Delta_1}{T_1}) > u\right\}} du\right] \\
&= D_\beta \mathbb{E}\left[\min(T_1 - \Delta_1, \Delta_1)\right].
\end{aligned}$$

Then the claim follows. We note that $\mathbb{E}\left[\min(T_1 - \Delta_1, \Delta_1)\right]$ is indeed finite, since

$$\mathbb{E}\left[\min(T_1 - \Delta_1, \Delta_1)\right] \leq \mathbb{E}\left[T_1 - \Delta_1\right] \leq \mathbb{E}\left[T_1 - \Delta^*\right]$$

and we can check by using the joint law of (Δ^*, T_1) that $\mathbb{E}\left[T_1 - \Delta^*\right] < \infty$. □

2.4.2 Length of the longest chord

For $\beta \in (1,2)$, we will find the law of the length of the longest chord in the β-stable lamination in the same way as in the Brownian triangulation. Let $2\pi L$ be the length of the minor arc corresponding to the longest chord in the β-stable lamination. In the β-stable lamination, almost surely no chord passes through the origin. Thus we still call the unique face that contains the origin the *centroid*, and the longest chord is still an edge of the centroid. Hence using the bijection obtained by Proposition 2.4.1 and the same arguments as in the proof of Lemma 2.3.1, we can prove that

$$\mathbb{P}(L < a) = \mathbb{E}\left[N(1-a)\right], \quad 0 < a < \frac{1}{2},$$

where $N(1-a)$ is the number of $(\psi_L, 1-a)$-large dislocations in Θ_β as in Lemma 2.3.1. This equation allows us to find the law of the length of the longest chord in the β-stable lamination.

Proposition 2.4.3. *For $\beta \in (1,2)$, let $2\pi L$ be the length of the minor arc associated with the longest chord in the β-stable lamination, then L has distribution function*

$$\mathbb{P}(L < a) = D_\beta C_\beta \mathbb{E}\left[T_1 \int_{\frac{\Delta_1}{aT_1}\vee 1}^{\frac{1}{1-a}} (1-x^{-1})^{-1/\beta} dx\right], \quad 0 < a < \frac{1}{2},$$

where $D_\beta = \frac{\beta^2\Gamma(2-\beta^{-1})}{\Gamma(2-\beta)}$, $C_\beta = \frac{1}{\beta\Gamma(1-\beta^{-1})}$.

Proof of Proposition 2.4.3. For $a \in (0, \frac{1}{2})$, as we have argued above, $\mathbb{P}(L \le a) = \mathbb{E}[N(1-a)]$. Let us calculate $\mathbb{E}[N(1-a)]$. Using the transformation in Section 2.2.2, we may regard $N(1-a)$ as the number of $(\psi_L, 1-a)$-large dislocations in $\mathbf{X}$, the homogeneous counterpart of Θ_β. Recall that $\psi_L(x, \mathbf{s}) = \min(x, 1-xs_1)$, $(x,\mathbf{s}) \in [0,1]\times\mathcal{S}$, it follows from Lemma 2.2.8 that

$$\mathbb{E}[N(1-a)] = \mathbb{E}\left[\int_0^\infty \sum_{i=1}^\infty \nu_\beta\left(\mathbf{s}\in\mathcal{S} : 1-s_1 > 1 - \frac{a}{X_i(r)}\right)\mathbf{1}_{\{X_i(r)>1-a\}}dr\right].$$

Using (2.2.5), (2.4.1) and Fubini's Theorem yields

$$\mathbb{E}[N(1-a)] = \int_0^{\log\frac{1}{1-a}} \mathrm{e}^x \int_{\mathcal{S}} \mathbf{1}_{\{1-s_1>1-a\mathrm{e}^x\}}\nu_\beta(d\mathbf{s})dU(x) = D_\beta\mathbb{E}\left[T_1\int_{(\log\frac{\Delta_1}{aT_1})\vee 0}^{\log\frac{1}{1-a}} \mathrm{e}^x dU(x)\right], \tag{2.4.6}$$

where U is the potential measure of the subordinator associated with $\mathbf{X}$ as in Lemma 2.2.3. From Lemma 2.2.3 and Equation (10) in [64], the Laplace transform of U is

$$\int_0^\infty e^{-px}dU(x) = \left(\int_{\mathcal{S}}\left(1-\sum_{i=1}^\infty s_i^{p+1}\right)\nu_\beta(d\mathbf{s})\right)^{-1} = \left(\beta\frac{\Gamma(p+1-1/\beta)}{\Gamma(p)}\right)^{-1}, \quad p>0.$$

Noticing that

$$\int_0^{+\infty} \mathrm{e}^{-px}(1-\mathrm{e}^{-x})^{-1/\beta}dx = \Gamma(1-1/\beta)\frac{\Gamma(p)}{\Gamma(p+1-1/\beta)}, \quad p>0,$$

we find the density of dU:

$$dU(x) = C_\beta(1-\mathrm{e}^{-x})^{-1/\beta}dx, \quad x \ge 0,$$

where $C_\beta = (\beta\Gamma(1-1/\beta))^{-1}$. Rewriting (2.4.6), we have

$$\mathbb{P}(L<a) = \mathbb{E}[N(1-a)] = D_\beta C_\beta \mathbb{E}\left[T_1\int_{\frac{\Delta_1}{aT_1}\vee 1}^{\frac{1}{1-a}}\left(1-x^{-1}\right)^{-1/\beta}dx\right].$$

□

2.5 Large dislocations in a dissipative fragmentation

In Theorem 2.2.6, if ν is dissipative, then $m = +\infty$ and the limit in the case when (H1) holds is 0. In this section, we obtain a finer result.

Throughout this section, we suppose that $\nu(\mathcal{S}) < \infty$. Then it is known that (Proposition 1.4 in [17]) the fragmentation $\mathbf{X}^{(\alpha)}$ has a discrete genealogical structure, which is roughly explained at the end of Section 2.2.4. Next let us assume the *Malthusian condition*:

[M] there exists $p^* > \underline{p}$, where $\underline{p} := \inf\{p > 0 : \int_{\mathcal{S}} (\sum_{i=1}^{\infty} s_i^p)\, \nu(d\mathbf{s}) < +\infty\}$, such that

$$\int_{\mathcal{S}} \Big(1 - \sum_{i=1}^{\infty} s_i^{p^*}\Big) \nu(d\mathbf{s}) = 0,$$

and for some constant $\gamma > 1$,

$$\int_{\mathcal{S}} \Big(\sum_{i=1}^{\infty} s_i^{p^*}\Big)^{\gamma} \nu(d\mathbf{s}) < \infty.$$

The constant p^* is called the *Malthusian parameter*. Write $\mathbf{X}^{(0)}$ for the homogeneous counterpart of $\mathbf{X}^{(\alpha)}$, then under the Malthusian condition,

$$W(t) := \sum_{i=1}^{\infty} X_i^{(0)}(t)^{p^*}, \quad t \geq 0$$

is a uniformly integrable martingale which converges to a non-negative random variable W_∞ (so $\mathbb{E}[W_\infty] = 1$), see for example Proposition 1.5 in [17]. The random variable W_∞ satisfies the distributional identity

$$W_\infty \stackrel{d}{=} \sum_{i=1}^{\infty} S_i^{p^*} W_\infty^i,$$

where $(S_i)_{i\in\mathbb{N}}$ has the distribution of $\nu(\mathcal{S})^{-1}\nu$, and $(W_\infty^i)_{i\geq 1}$ are independent copies of W_∞, also independent of $(S_i)_{i\in\mathbb{N}}$. A more natural way to describe W_∞ is by using the the genealogical structure of the fragmentation $\mathbf{X}^{(\alpha)}$ and a genealogical martingale, so-called *the intrinsic martingale*; see [23]. So W_∞ is usually called *the terminal value of the intrinsic martingale of* $\mathbf{X}^{(\alpha)}$.

Theorem 2.5.1. *Consider a self-similar fragmentation $(\mathbf{X}^{(\alpha)}(t))_{t\geq 0} = (X_1^{(\alpha)}(t), X_2^{(\alpha)}(t), \ldots)_{t\geq 0}$ of index $\alpha \in \mathbb{R}$ with no erosion and non-lattice (Definition 2.2.4) dislocation measure ν. Suppose that $\nu(\mathcal{S}) < \infty$ and that the Malthusian condition* [M] *holds with Malthusian parameter $p^* > 0$. Let $\varphi : \mathcal{S} \to [0, \infty)$ be bounded and $b > 0$, and write $N(\epsilon)$ for the total number of $(\varphi x^b, \epsilon)$-large* *dislocations in $\mathbf{X}^{(\alpha)}$. Then we have that*

$$\lim_{\epsilon\to 0}\epsilon^{\frac{p^*}{b}}N(\epsilon)=\frac{\int_0^\infty y^{(p^*-1)}g(y^b)dy}{m(p^*)}W_\infty,\quad in\ L^1(\mathbb{P}),$$

where W_∞ is the terminal value of the intrinsic martingale of $\mathbf{X}^{(\alpha)}$, the function g is defined by

$$g(u):=\nu\left(\mathbf{s}\in\mathcal{S}:\varphi(\mathbf{s})>u\right),\quad u>0,$$

and

$$m(p^*):=\int_{\mathcal{S}}\sum_{n=1}^\infty s_n^{p^*}\log(s_n^{-1})\nu(d\mathbf{s}).$$

The proof is inspired by the proof in Theorem 1 in [24].

Proof. It suffices to prove the result for the homogeneous case $\mathbf{X}^{(0)}$. We will associate $\mathbf{X}^{(0)}$ with a Crump-Mode-Jagers branching process (see [50]) constructed as follows. Let $\lambda=+\infty$ and ξ be a point process on $(0,\infty)$ whose atoms are $(-\log s_1,-\log s_2,\ldots)$, where $\mathbf{s}\in\mathcal{S}$ follows the probability measure $\nu(\mathcal{S})^{-1}\nu$. Each individual $u\in\mathcal{U}$ is labeled with an pair (λ_u,ξ_u), an i.i.d. copy of (λ,ξ). We may regard λ_u as the lifetime of the individual u and ξ_u as the points of birth times of the children of u relative to u's own birth time. Further, denote the birth time of u by σ_u.

By our construction, the tree $\mathcal{U}$ marked by $m_u:=\mathrm{e}^{\sigma_u}$ for each $u\in\mathcal{U}$ has the same law as the genealogical tree of the fragmentation $\mathbf{X}^{(0)}$ (see Section 1.1.3). For $t\geq 0$, let $l(t)\subset\mathcal{U}$ denote the individuals that are born after t whose parents were born before t, then

$$Y_t:=\sum_{u\in l(t)}\mathrm{e}^{-\sigma_u},\qquad t\geq 0$$

is a uniformly integrable martingale, whose limit as $t\to\infty$ coincides with W_∞; see [50].

Fix a constant $M>|\varphi|$. For each $u\in\mathcal{U}$, we define a function $\phi_u:\mathbb{R}\to\mathbb{R}_+$ by

$$\phi_u(t)=\mathbf{1}_{\left\{\varphi(\mathbf{s}_u)>M\mathrm{e}^{-bt}\right\}},\qquad t\in\mathbb{R}.$$

Clearly we have $\phi_u(t)=0$ if $t\leq 0$, and the triples (λ_u,ξ_u,ϕ_u) are still i.i.d. across individuals $u\in\mathcal{U}$. For any $p_0\in(\underline{p},p^*)$, there are

$$\mathbb{E}\left[\int_{(0,\infty)}\mathrm{e}^{-p_0t}\xi(dt)\right]=\nu(\mathcal{S})^{-1}\int_{\mathcal{S}}\sum_{i=1}^\infty s_i^{p_0}\nu(d\mathbf{s})<\infty,$$

and

$$\mathbb{E}\left[\sup_{t\geq 0}e^{-p_0t}\phi_\emptyset(t)\right]=\nu(\mathcal{S})^{-1}\int_{\mathcal{S}}\left(\frac{\varphi(\mathbf{s})}{M}\right)^{\frac{p_0}{b}}\nu(d\mathbf{s})<\infty.$$

Thus according to Theorem 6 in [57] (also see [68]), the following convergence holds in $L^1(\mathbb{P})$

$$\lim_{t\to\infty} \mathrm{e}^{-p^*t} \sum_{u\in\mathcal{U}} \phi_u(t-\sigma_u) = \frac{\int_0^\infty \mathrm{e}^{-p^*t}\mathbb{E}\left[\phi_\emptyset(t)\right] dt}{\int_0^\infty t\mathrm{e}^{-p^*t} d\mu(t)} W_\infty = \frac{\int_0^\infty g(y^b) y^{(p^*-1)} dy}{m(p^*)} M^{-\frac{p^*}{b}} W_\infty, \quad (2.5.1)$$

where μ is the reproduction intensity defined by

$$\mu(t) := \mathbb{E}\left[\xi([0,t])\right] = \int_{\mathcal{S}} \sum_{n=1}^{\infty} \mathbf{1}_{\{s_n \geq \mathrm{e}^{-t}\}} \nu(d\mathbf{s}),$$

and the last equality follows from direct calculation. On the other hand, for every $\epsilon > 0$, let $\mathrm{e}^{-t} = (\frac{\epsilon}{M})^{\frac{1}{b}}$, then

$$\mathrm{e}^{-p^*t} \sum_{u\in\mathcal{U}} \phi_u(t-\sigma_u) = \epsilon^{\frac{p^*}{b}} M^{-\frac{p^*}{b}} N(\epsilon).$$

Hence the claim follows from (2.5.1). □

Let us discuss an application of Theorem 2.5.1 in the packing problems, which bear an interesting connection with modeling of communication networks, see [9] and references therein.

Packing intervals We consider the self-similar sequential interval packing process studied in [9]. We begin with the interval $[0,1]$ and pack the first interval $[X, 1-Y]$ after a time of exponential law, where the pair (X,Y) has joint law F on $\Delta = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1, x+y \leq 1\}$. Thus the interval $[0,1]$ splits into three intervals: $[0,X]$ and $[1-Y,1]$ are two *gaps* and $[X,1-Y]$ is the *packed interval*. When a gap of length x is created, after a time of exponential law with parameter x^β with $\beta \in \mathbb{R}$, it splits into three pieces, a packed interval of length $x(1-X'-Y')$ in the middle and two gaps of length xX' and xY', where (X',Y') is an independent copy of (X,Y). It is plain that the lengths of gaps form a self-similar fragmentation with index of self-similarity β and no erosion, whose dislocation measure ν is the joint law of the order statistics (but by decreasing order) of (X,Y). Denote this fragmentation by $\mathbf{X}$.

Since we are only interested in the lengths of gaps and intervals, but not in their arrangements, without loss of generality, X and Y are exchangeable, in particular they have the same marginal distribution denoted by G_X. We also assume G_X is absolutely continuous. According to [9], the Malthusian parameter p^* is the unique solution in $(0,1]$ such that

$$2\mathbb{E}\left[X^{p^*}\right] = 1.$$

We easily check that **[M]** holds, as for any constant $\gamma > 1$,

$$\int_\Delta \left(x^{p^*} + y^{p^*}\right)^\gamma F(dxdy) \leq \int_\Delta 2^\gamma F(dxdy) < 2^\gamma.$$

Let $N(\epsilon)$ be the number of packed intervals with length greater than ϵ, then it is equal to the number of dislocations whose mark $(x, \mathbf{s})$ satisfies $(1-s_1-s_2)x > \epsilon$. Write G_Z for the marginal distribution of $Z := 1-X-Y$, we have that

$$\int_0^\infty g(u)u^{(p^*-1)}du = \int_0^\infty \int_\Delta \mathbf{1}_{\{(1-x-y)>u\}} dF(x,y)u^{(p^*-1)}du = \frac{1}{p^*}\int_{(0,1)} z^{p^*} G_Z(dz) = \mathbb{E}\left[\frac{Z^{p^*}}{p^*}\right].$$

and

$$m(p^*) := 2\int_{(0,1)} x^{p^*} \log(x^{-1}) G_X(dx) = 2\mathbb{E}\left[-X^{p^*}\log X\right].$$

Using Theorem 2.5.1, we conclude the following statement.

Proposition 2.5.2. *In a packing model with any index of self similarity, lengths of gaps X and Y and length of packed interval $Z = 1-X-Y$, let W_∞ be the terminal value of the intrinsic martingale and $N(\epsilon)$ be the number of packed intervals with length greater than $\epsilon > 0$, then*

$$\lim_{\epsilon\to 0} \epsilon^{p^*} N(\epsilon) = \frac{\mathbb{E}\left[Z^{p^*}\right]}{2p^*\mathbb{E}\left[-X^{p^*}\log X\right]} W_\infty \quad \textit{in } L^1(\mathbb{P}).$$

Combining Theorem 6 in [9] and Corollary 6 in [23], we note that $W_\infty \geq 0$ is the unique solution of the distributional equation

$$W_\infty \overset{d}{=} X^{p^*} W_\infty^1 + Y^{p^*} W_\infty^2,$$

where W_∞^1 and W_∞^2 are independent copies of W_∞, also independent of (X, Y).

We next look at the special case when the packing law is given by

$$dF(x,y) = \beta(\beta-1)(1-x-y)^{\beta-2}dxdy, \quad (x,y) \in \Delta, \tag{2.5.2}$$

with $\beta \geq 2$ an integer. Then the marginal laws of X and Z have respect densities

$$G_X(dx) = \beta(1-x)^{\beta-1}dx, \quad x \in (0,1) \qquad \text{and} \qquad G_Z(dz) = \beta(\beta-1)z^{\beta-2}(1-z)dz, \quad z \in (0,1).$$

Therefore, the Malthusian parameter p^* satisfies

$$\frac{\Gamma(p^*+1)\Gamma(\beta+1)}{\Gamma(p^*+\beta+1)} = \frac{\beta!}{\prod_{i=1}^{\beta}\frac{1}{p^*+i}} = \frac{1}{2}.$$

Using this identity we deduce that

$$\lim_{\epsilon\to 0} \epsilon^{p^*} N(\epsilon) = \frac{\beta(\beta-1)}{p^*(p^*+\beta)(p^*+\beta-1)} \left(\sum_{i=1}^{\beta} \frac{1}{p^*+i}\right)^{-1} W_\infty \quad \text{in } L^1(\mathbb{P}).$$

In particular, if $\beta = 2$, which means the packed interval in a given gap is the interval between two independent uniform random variables in this gap, then the Malthusian parameter is $p^* = \frac{\sqrt{17}-3}{2}$ and the limit is $\frac{2}{p^*(2p^*+3)} W_\infty$.

Packing rectangles Motivated by the interval packing process, we consider a self-similar rectangle packing process in $[0,1]^2$, which can be roughly viewed as a random *Sierpinski carpet*. Specifically, we still consider the measure F on Δ as in the interval packing, and let $(X_1, Y_1) \times (X_2, Y_2)$ have distribution $F \otimes F$. We next set the dynamics as follows. Beginning at time zero, we pack the square $[X_1, 1 - Y_1] \times [X_2, 1 - Y_2]$ in $[0,1]^2$ after an exponential time. Then the lines $X_1 \times [0,1]$, $Y_1 \times [0,1]$, $X_2 \times [0,1]$, $Y_2 \times [0,1]$ divide the square $[0,1]^2$ into nine smaller rectangles: one packed rectangle in the center and eight gap rectangles around. When a gap rectangle $[x_1, 1 - y_1] \times [x_2, 1 - y_2]$ of area $m = (1 - y_1 - x_1)(1 - y_2 - x_2)$ is created, it waits for a mean m^β exponential time to split into 9 smaller rectangles: a packed rectangle

$$[x_1 + X_1'(y_1 - x_1), x_1 + Y_1'(y_1 - x_1)] \times [x_2 + X_2'(y_2 - x_2), x_2 + Y_2'(y_2 - x_2)]$$

where $(X_1', Y_1') \times (X_2', Y_2')$ is an independent copy of $(X_1, Y_1) \times (X_2, Y_2)$, and eight gaps around.

Then the areas of the gaps form a self-similar fragmentation with self-similarity index β, and dislocation measure is determined by F. Let (X, Y) be a random variable of law F, and let $Z := (1 - X - Y)$. Then the Malthusian condition is satisfied with Malthusian parameter satisfying

$$1 = 4\mathbb{E}\left[X^{p^*}\right]^2 + 4\mathbb{E}\left[X^{p^*}\right]\mathbb{E}\left[Z^{p^*}\right].$$

Denote by W_∞ the terminal value of the intrinsic martingale, the following result follows immediately from Theorem 2.5.1.

Proposition 2.5.3. *Let $N(\epsilon)$ be the number of packed squares whose area is greater than $\epsilon > 0$. Then we have*

$$\lim_{\epsilon \to 0} \epsilon^{p^*} N(\epsilon) = \frac{C_1}{C_2} W_\infty, \quad \textit{in } L^1(\mathbb{P}),$$

where $C_1 := \frac{1}{p^*}\mathbb{E}\left[Z^{p^*}\right]^2$ and

$$\begin{aligned} C_2 :=& 8\mathbb{E}\left[X^{p^*}\log(X^{-1})\right]\mathbb{E}\left[X^{p^*}\right] \\ &+ 4\mathbb{E}\left[Z^{p^*}\log(Z^{-1})\right]\mathbb{E}\left[X^{p^*}\right] + 4\mathbb{E}\left[X^{p^*}\log(X^{-1})\right]\mathbb{E}\left[Z^{p^*}\right]. \end{aligned}$$

We finally give the explicit results for F in form of (2.5.2). Denote

$$S := \sum_{i=1}^{\beta} \frac{1}{p^* + i}, \quad P := \frac{\Gamma(p^*+1)\Gamma(\beta+1)}{\Gamma(p^*+\beta+1)}, \quad A := \frac{\beta(\beta-1)}{(p^*+\beta)(p^*+\beta-1)},$$

then the Malthusian parameter p^* follows

$$1 = 4P(P + A) = 4P\left(P + \frac{\beta(\beta-1)}{(p^*+\beta)(p^*+\beta-1)}\right)$$

and by calculation we get that $C_1 = \frac{1}{p^*}A^2$ and

$$C_2 = 2S - 4PA\sum_{i=1}^{\beta-2}\frac{1}{p^*+i} = 2S - 4PA\left(\frac{1}{p^*+\beta} + \frac{1}{p^*+\beta-1} - S\right).$$

In particular when $\beta = 2$, there is $(p^*+1)^2(p^*+2)^2 = 32$ with $p^* = \frac{\sqrt{16\sqrt{2}+1}-3}{2}$, and the limit becomes

$$\frac{\sqrt{2}}{4}\frac{1}{p^*(2p^*+3)}W_\infty.$$

2.6 Proofs of the technical statements

In the section we prove Lemma 2.2.12 and Lemma 2.2.13.

Proof of Lemma 2.2.12. Given V, a uniform random variable in $(0,1)$ independent of $\mathbf{X}$, we obtain the fragment tagged by V in the same way as in Section 2.2.3, and denote its mass by χ. Set

$$\tilde{A}_t(\epsilon) = \int_0^t \chi(s)^{-1} f_1(\epsilon^{-\frac{1}{b}}\chi(s))ds, \quad t \geq 0.$$

Note that $f_1(0) = 0$ and by convention $0^{-1} \times 0 = 0$. Let $\tilde{A}(\epsilon) := \lim_{t\to\infty}\tilde{A}_t(\epsilon)$. Given another random variable V', uniform in $(0,1)$ and independent of V and $\mathbf{X}$, we define χ', $\tilde{A}'_t(\epsilon)$ and $\tilde{A}'(\epsilon)$ in the same way. Using (2.2.5) yields

$$\mathbb{E}\left[\tilde{A}_t(\epsilon)\right] = \mathbb{E}\left[\tilde{A}'_t(\epsilon)\right] = \mathbb{E}\left[\int_0^t \sum_{i=1}^{\infty} f_1(\epsilon^{-\frac{1}{b}}X_i(r))dr\right].$$

Since (V, V') has uniform distribution in $[0,1]^2$, independent of $\mathcal{F}_t$, we deduce the distribution of $(\chi(t), \chi'(t))$ conditionally on $\mathcal{F}_t$:

$$\mathbb{P}\left[(\chi(t), \chi'(t)) = (X_i(t), X_j(t)) \;\middle|\; \mathcal{F}_t\right] = X_i(t)X_j(t), \quad \forall i, j \in \mathbb{N}.$$

By standard calculation, there is

$$\mathbb{E}\left[\tilde{A}_t(\epsilon)\tilde{A}'_t(\epsilon)\right] = \mathbb{E}\left[\left(\int_0^t \sum_{i=1}^{\infty} f_1(\epsilon^{-\frac{1}{b}}X_i(r))dr\right)^2\right].$$

Letting $t \to +\infty$ and using the monotone convergence theorem, we obtain

$$\mathbb{E}\left[\tilde{A}(\epsilon)\right] = \mathbb{E}\left[A(\epsilon)\right] \quad \text{and} \quad \mathbb{E}\left[\tilde{A}(\epsilon)\tilde{A}'(\epsilon)\right] = \mathbb{E}\left[A(\epsilon)^2\right].$$

Let T be the first instant when V and V' belong to two different intervals, with respective lengths $\chi(T)$ and $\chi'(T)$. Set

$$S := \tilde{A}_T(\epsilon), \quad R := \int_T^\infty \chi(t)^{-1} f_1(\epsilon^{-\frac{1}{b}}\chi(t))dt, \quad R' := \int_T^\infty \chi'(t)^{-1} f_1(\epsilon^{-\frac{1}{b}}\chi'(t))dt.$$

For conciseness we did not indicate their dependence on ϵ. Then $\tilde{A}(\epsilon) = S+R$ and $\tilde{A}'(\epsilon) = S+R'$. Hence

$$\mathbb{E}\left[A(\epsilon)^2\right] = \mathbb{E}\left[S^2 + S(R+R') + RR'\right].$$

We will calculate each term. Let us begin with $\mathbb{E}\left[S(R+R')\right]$. By Markov property,

$$\begin{aligned}
&\mathbb{E}\left[S(R+R') \;\middle|\; S, \chi(T), \chi'(T)\right] \\
&\quad = S\left(\chi(T)^{-1}\,\mathbb{E}\left[A(x^{-b}\epsilon)\right]\Big|_{x=\chi(T)} + \chi'(T)^{-1}\,\mathbb{E}\left[A((x')^{-b}\epsilon)\right]\Big|_{x'=\chi'(T)}\right).
\end{aligned}$$

As $A(\epsilon) = 0$ for $\epsilon > |\varphi|$, from Lemma 2.2.11 it is clear that there exists $\bar{c}_A > 0$ such that for every $\epsilon > 0$,

$$\epsilon^{\frac{1}{b}}\mathbb{E}\left[A(\epsilon)\right] \le \bar{c}_A. \tag{2.6.1}$$

Thus

$$\epsilon^{\frac{2}{b}}\mathbb{E}\left[S(R+R')\right] \le 2\bar{c}_A\epsilon^{\frac{1}{b}}\mathbb{E}\left[S\right]. \tag{2.6.2}$$

We now deal with $\mathbb{E}\left[RR'\right]$. Let $\bar{\chi}$ and $\bar{\chi}'$ be two independent copies of χ. By the branching property and the Markov property, we have that

$$\begin{aligned}
&\mathbb{E}\left[RR' \;\middle|\; \chi(T), \chi'(T)\right] \\
&= \mathbb{E}\left[\int_0^\infty (x\bar{\chi}(s))^{-1} f_1(x\epsilon^{-\frac{1}{b}}\bar{\chi}(s))ds \int_0^\infty (x'\bar{\chi}'(r))^{-1} f_1(x'\epsilon^{-\frac{1}{b}}\bar{\chi}'(r))dr\right]\Bigg|_{x=\chi(T),x'=\chi'(T)} \\
&= \chi(T)^{-1}\chi'(T)^{-1}\,\mathbb{E}\left[A(x^{-b}\epsilon)\right]\Big|_{x=\chi(T)}\,\mathbb{E}\left[A((x')^{-b}\epsilon)\right]\Big|_{x'=\chi'(T)}.
\end{aligned}$$

It then follows from (2.6.1) that for every $\epsilon > 0$,

$$\epsilon^{\frac{2}{b}}\mathbb{E}\left[RR' \;\middle|\; \chi(T), \chi'(T)\right] \le (\bar{c}_A)^2.$$

We deduce from Lemma 2.2.11 that

$$\lim_{\epsilon\to 0}\epsilon^{\frac{2}{b}}\mathbb{E}\left[RR' \;\middle|\; \chi(T), \chi'(T)\right] = \left(\frac{1}{m}\int_0^\infty g(u^b)du\right)^2.$$

Taking expectation in the last limit and using the dominated convergence theorem, we have that

$$\lim_{\epsilon\to 0}\epsilon^{\frac{2}{b}}\mathbb{E}\left[RR'\right]=\left(\frac{1}{m}\int_0^\infty g(u^b)du\right)^2. \tag{2.6.3}$$

We next calculate $\mathbb{E}\left[S^2\right]$. It is clear that

$$\begin{aligned}\mathbb{E}\left[S^2\right]&=2\int_{(0,\infty)^2}\mathbb{E}\left[\mathbf{1}_{\{s<t\}}\mathbf{1}_{\{s<T\}}\mathbf{1}_{\{t<T\}}\chi(s)^{-1}\chi(t)^{-1}f_1(\epsilon^{-\frac{1}{b}}\chi(s))f_1(\epsilon^{-\frac{1}{b}}\chi(t))\right]dsdt\\&\leq 2\int_{(0,\infty)^2}\mathbb{E}\left[\mathbf{1}_{\{s<t\}}\mathbf{1}_{\{s<T\}}\chi(s)^{-1}\chi(t)^{-1}f_1(\epsilon^{-\frac{1}{b}}\chi(s))f_1(\epsilon^{-\frac{1}{b}}\chi(t))\right]dsdt.\end{aligned}$$

Let $\bar{\chi}$ be an independent copy of χ. By the Markov property, the last quantity equals

$$2\int_0^\infty ds\mathbb{E}\left[\mathbf{1}_{\{s<T\}}\chi(s)^{-1}f_1(\epsilon^{-\frac{1}{b}}\chi(s))\;\mathbb{E}\left[\int_0^\infty x^{-1}\bar{\chi}(u)^{-1}f_1(\epsilon^{-\frac{1}{b}}x\bar{\chi}(u))du\right]\Bigg|_{x=\chi(s)}\right].$$

Multiplying by $\epsilon^{\frac{2}{b}}$ and using (2.6.1), we obtain that

$$\epsilon^{\frac{2}{b}}\mathbb{E}\left[S^2\right]\leq 2\epsilon^{\frac{1}{b}}\int_0^\infty ds\mathbb{E}\left[\mathbf{1}_{\{s<T\}}\chi(s)^{-1}f_1(\epsilon^{-\frac{1}{b}}\chi(s))\bar{c}_A\right]=2\bar{c}_A\epsilon^{\frac{1}{b}}\mathbb{E}\left[S\right]. \tag{2.6.4}$$

We finally look at $\mathbb{E}[S]$. For $t>0$, write $I_{n(t)}(t)$ for the interval fragment containing V at time t as in Section 2.2.3, thus $|I_{n(t)}(t)|=\chi(t)$. Since V' is independent of $I_{n(t)}(t)$,

$$\begin{aligned}&\mathbb{E}\left[\chi(t)^{-1}f_1(\epsilon^{-\frac{1}{b}}\chi(t))\mathbf{1}_{\{t\leq T\}}\;\middle|\;I_{n(t)}(t)\right]\\&\quad=\chi(t)^{-1}f_1(\epsilon^{-\frac{1}{b}}\chi(t))\mathbb{E}\left[\mathbf{1}_{\{V'\in I_{n(t)}(t)\}}\;\middle|\;I_{n(t)}(t)\right]=f_1(\epsilon^{-\frac{1}{b}}\chi(t)).\end{aligned}$$

Therefore, we have

$$\mathbb{E}\left[S\right]=\int_0^\infty\mathbb{E}\left[\mathbb{E}\left[\chi(t)^{-1}f_1(\epsilon^{-\frac{1}{b}}\chi(t))\mathbf{1}_{\{t\leq T\}}\;\middle|\;I_{n(t)}(t)\right]\right]dt=\int_0^\infty\mathbb{E}\left[f_1(\epsilon^{-\frac{1}{b}}\chi(t))\right]dt.$$

Multiplying by $\epsilon^{\frac{1}{b}}$ and using (2.2.13) and then Lemma 2.2.3, we have that

$$\epsilon^{\frac{1}{b}}\int_0^\infty\mathbb{E}\left[f_1(\epsilon^{-\frac{1}{b}}\chi(t))\right]dt\leq\epsilon^{\frac{1-ab}{b}}\bar{c}\int_0^\infty\mathbb{E}\left[\mathrm{e}^{-ab\xi(t)}\right]dt=\bar{c}\epsilon^{\frac{1-ab}{b}}\frac{1}{\Phi(ab)}.$$

Since $1>ab$, the right-hand side tends to 0 as $\epsilon\to 0$. Hence

$$\lim_{\epsilon\to 0}\epsilon^{\frac{1}{b}}\mathbb{E}\left[S\right]=\lim_{\epsilon\to 0}\epsilon^{\frac{1}{b}}\mathbb{E}\left[\int_0^\infty f_1(\epsilon^{-\frac{1}{b}}\chi(t))dt\right]=0. \tag{2.6.5}$$

Combining (2.6.2),(2.6.3), (2.6.4) and (2.6.5), we prove this lemma. □

Proof of Lemma 2.2.13. Define the function $\bar{\psi}$ by $\bar{\psi}(x, \mathbf{s}) := \psi(x, \mathbf{s})\mathbf{1}_{\{x>\frac{1}{2}\}}$, $(x, \mathbf{s}) \in [0, 1] \times \mathcal{S}$. Then $\bar{N}(\epsilon)$ is the number of $(\bar{\psi}, \epsilon)$-large dislocations, and the random variable defined as in (2.2.9) is

$$\bar{A}(\epsilon) := \int_0^\infty \sum_{i=1}^\infty f_1(X_i(r)\epsilon^{-\frac{1}{b}})\mathbf{1}_{\{X_i(r)>\frac{1}{2}\}}dr.$$

Applying (2.2.5), we have

$$\mathbb{E}\left[\bar{A}(\epsilon)\right] = \int_0^\infty \mathbb{E}\left[\chi(r)^{-1}f_1(\chi(r)\epsilon^{-\frac{1}{b}})\mathbf{1}_{\{\chi(r)>\frac{1}{2}\}}\right]dr \le 2\mathbb{E}\left[\int_0^\infty f_1(\chi(r)\epsilon^{-\frac{1}{b}})dr\right].$$

Multiplying by $\epsilon^{\frac{1}{b}}$ and using (2.6.5), we have

$$\lim_{\epsilon\to 0}\epsilon^{\frac{1}{b}}\mathbb{E}\left[\bar{A}(\epsilon)\right] = 0. \tag{2.6.6}$$

Next we study $\mathbb{E}\left[\bar{A}(\epsilon)^2\right]$. Using the same notations as in the proof of Lemma 2.2.12, we set

$$\begin{aligned}
\bar{S} &:= \int_0^T \chi(t)^{-1}f_1(\epsilon^{-\frac{1}{b}}\chi(t))\mathbf{1}_{\{\chi(t)>\frac{1}{2}\}}dt,\\
\bar{R} &:= \int_T^\infty \chi(t)^{-1}f_1(\epsilon^{-\frac{1}{b}}\chi(t))\mathbf{1}_{\{\chi(t)>\frac{1}{2}\}}dt,\\
\bar{R}' &:= \int_T^\infty \chi'(t)^{-1}f_1(\epsilon^{-\frac{1}{b}}\chi'(t))\mathbf{1}_{\{\chi'(t)>\frac{1}{2}\}}dt.
\end{aligned}$$

Thus similarly there is

$$\mathbb{E}\left[\bar{A}(\epsilon)^2\right] = \mathbb{E}\left[\bar{S}^2 + \bar{S}(\bar{R}+\bar{R}') + \bar{R}\bar{R}'\right].$$

On the one hand, letting $\bar{\chi}$ and $\bar{\chi}'$ be two independent copies of χ, we see from the branching property and the Markov property that

$$\begin{aligned}
&\mathbb{E}\left[\bar{R}\bar{R}' \;\middle|\; \chi(T), \chi'(T)\right]\\
&= \mathbb{E}\Big[\int_0^\infty (x\bar{\chi}(s))^{-1}\mathbf{1}_{\{x\bar{\chi}(s)>\frac{1}{2}\}}f_1(x\epsilon^{-\frac{1}{b}}\bar{\chi}(s))ds\\
&\qquad\qquad \times \int_0^\infty (x'\bar{\chi}'(r))^{-1}\mathbf{1}_{\{x'\bar{\chi}'(r)>\frac{1}{2}\}}f_1(x'\epsilon^{-\frac{1}{b}}\bar{\chi}'(r))dr\Big]\Big|_{x=\chi(T),x'=\chi'(T)}\\
&\le \mathbb{E}\Big[\int_0^\infty (x\bar{\chi}(s))^{-1}\mathbf{1}_{\{\bar{\chi}(s)>\frac{1}{2}\}}f_1(x\epsilon^{-\frac{1}{b}}\bar{\chi}(s))ds\\
&\qquad\qquad \times \int_0^\infty (x'\bar{\chi}'(r))^{-1}\mathbf{1}_{\{\bar{\chi}'(r)>\frac{1}{2}\}}f_1(x'\epsilon^{-\frac{1}{b}}\bar{\chi}'(r))dr\Big]\Big|_{x=\chi(T),x'=\chi'(T)}\\
&= \chi(T)^{-1}\chi'(T)^{-1}\,\mathbb{E}\left[\bar{A}(x^{-b}\epsilon)\right]\Big|_{x=\chi(T)}\,\mathbb{E}\left[\bar{A}((x')^{-b}\epsilon)\right]\Big|_{x'=\chi'(T)}.
\end{aligned}$$

Then it follows from (2.6.6) that

$$\lim_{\epsilon\to 0}\epsilon^{\frac{2}{b}}\mathbb{E}\left[\bar{R}\bar{R}' \;\middle|\; \chi(T), \chi'(T)\right] = 0.$$

Taking expectation and using the dominated convergence theorem, we see that

$$\lim_{\epsilon\to 0}\epsilon^{\frac{2}{b}}\mathbb{E}\left[\bar{R}\bar{R}'\right] = 0.$$

On the other hand, we observe from (2.6.2) and (2.6.4) that

$$\epsilon^{\frac{2}{b}}\mathbb{E}\left[\bar{S}^2 + \bar{S}(\bar{R}+\bar{R}')\right] \le \epsilon^{\frac{2}{b}}\mathbb{E}\left[S^2 + S(R+R')\right] \le 4\bar{c}_A\epsilon^{\frac{1}{b}}\mathbb{E}\left[S\right],$$

then it follows from (2.6.5) that

$$\lim_{\epsilon\to 0}\epsilon^{\frac{2}{b}}\mathbb{E}\left[\bar{S}^2 + \bar{S}(\bar{R}+\bar{R}')\right] = 0.$$

So we conclude

$$\lim_{\epsilon\to 0}\epsilon^{\frac{2}{b}}\mathbb{E}\left[\bar{A}(\epsilon)^2\right] = \lim_{\epsilon\to 0}\epsilon^{\frac{2}{b}}\mathbb{E}\left[\bar{S}^2 + \bar{S}(\bar{R}+\bar{R}') + \bar{R}\bar{R}'\right] = 0.$$

The claim thus follows from Corollary 2.2.10. □

Chapter 3

Growth-fragmentation processes and bifurcators

This chapter is mainly based on [76].

Markovian growth-fragmentation processes introduced by Bertoin model a system of growing and splitting cells in which the size of a typical cell evolves as a Markov process X without positive jumps. We find that two growth-fragmentation processes associated respectively with two processes X and Y (with different laws) may have the same distribution, if (X, Y) is a *bifurcator*, roughly speaking, which means that they coincide up to a bifurcation time and then evolve independently. Using this criterion, we deduce that the law of a self-similar growth-fragmentation is determined by a cumulant function κ and its index of self-similarity.

3.1 Introduction

We consider the family of *Markovian growth-fragmentation processes* introduced by Bertoin [20], see also [30, 31, 39, 46] for related works. This stochastic model describes the evolution of a particle system, in which each particle may grow or decay gradually and split randomly into smaller pieces, independently of the other particles. It is convenient to describe it in terms of a cell population. The size of a typical cell evolves as a Markov process $X = (X(t), t \geq 0)$ with values in $[0, \infty)$, with càdlàg path and only negative jumps. The process X also encodes the relationship between cell size and cell replication: at each jump time $t \geq 0$ of X with $\Delta X(t) = X(t) - X(t-) < 0$, a "daughter" cell with initial size $-\Delta X(t)$ is born, and the "mother" is still alive after this cell replication. Each daughter follows the same dynamics as the mother and evolves independently of the other cells. Starting at time 0 from a single cell with size $x > 0$, we construct in this way a population of cells and thus define a process $\mathbf{X} = (\mathbf{X}(t), t \geq 0)$, where $\mathbf{X}(t)$ denotes the multiset (that allows multiple instances of the elements) of the sizes of the cells alive at time $t \geq 0$. The process $\mathbf{X}$ is called a *(Markovian) growth-fragmentation process starting from x associated with the cell process X.*

By construction, the law of $\mathbf{X}$ is determined by the law of X, however, growth-fragmentations driven by cell processes with different laws may have the same distribution. A first instance of such processes appears in Pitman and Winkel [70] with X the exponential of the negative of a pure-jump subordinator (so-called *fragmenter* in [70]). The main purpose of this work is therefore to provide a sufficient condition for growth-fragmentations driven by different cell processes to have the same distribution. Our main result can be informally described as follows:

If there exists a coupling of (the distributions of) two cell processes X and Y which is a **bifurcator***, in the sense that they almost surely coincide for a strictly positive time and evolve independently afterwards, then under some mild technical conditions, the growth-fragmentations driven respectively by X and Y have the same finite-dimensional distributions.*

This will be stated rigorously in Theorem 3.3.9. The idea of bifurcator also goes back to [70], which provides an explicit construction of bifurcators of fragmenters, as well as a characterization of the laws of all bifurcators of fragmenters.

Therefore, to give a sufficient condition for two growth-fragmentations to have the same distribution, it suffices to understand when two cell processes can be coupled to form a bifurcator (in other words, when there exists a bifurcator whose two marginal distributions are the respective laws of these two cell processes). We do not have a complete answer to this question in general, however, we investigate a study of bifurcators for positive self-similar Markov processes, which further allows us to characterize the laws of growth-fragmentations driven by self-similar processes, so-called *self-similar growth-fragmentation processes.*

Self-similar growth-fragmentations have been previously studied in [20] and have interesting applications: this model is connected with certain growth-fragmentation equations, see [27]; besides, a distinguished case of self-similar growth-fragmentation appears as the re-scaled limit of the lengths of the cycles obtained by slicing random Boltzmann triangulations with a simple boundary at heights, see [22].

In order to state our results, let us recall some basic facts about Lévy processes, which are closely related to self-similar Markov processes; see e.g. [12, 58]. Let ξ be a Lévy process with no positive jumps, which is often referred to as a *spectrally negative Lévy process (SNLP).* The SNLP ξ is possibly killed at some independent exponential time. The distribution of ξ is characterized by its Laplace exponent $\Phi : [0, \infty) \to \mathbb{R}$:

$$\mathbb{E}\left[\mathrm{e}^{q\xi(t)}\right] = \mathrm{e}^{\Phi(q)t}, \qquad \text{for all } q, t \geq 0.$$

It is well-known that the convex function Φ can be expressed by the Lévy-Khintchine formula

$$\Phi(q) = -k + \frac{1}{2}\sigma^2 q^2 + cq + \int_{(-\infty,0)} \left(\mathrm{e}^{qz} - 1 + q(1 - \mathrm{e}^{z})\right) \Lambda(dz), \quad q \geq 0, \tag{3.1.1}$$

where $k \geq 0$ is the killing rate, $\sigma \geq 0$, $c \in \mathbb{R}$ and the Lévy measure Λ on $(-\infty, 0)$ satisfies

$$\int_{(-\infty,0)} (|z|^2 \wedge 1)\Lambda(dz) < \infty. \tag{3.1.2}$$

Then we say ξ is a SNLP with characteristics (σ, c, Λ, k). We also introduce $\kappa\colon [0,\infty) \to (-\infty, \infty]$ which plays an important role in this work:

$$\kappa(q) := \Phi(q) + \int_{(-\infty,0)} (1 - \mathrm{e}^z)^q \Lambda(dz), \quad q \geq 0. \tag{3.1.3}$$

So $\kappa \geq \Phi$. Note that κ is convex and $\kappa(q) < \infty$ for all $q \geq 2$ because of (3.1.2). We stress that κ does not characterize the law of ξ, see Lemma 3.2.1.

Let $X^{(0)} := \exp(\xi)$, and we write by convention $X^{(0)}(t) = \partial$ if ξ is killed before t, where ∂ denotes a cemetery point. Then the process $X^{(0)}$ is called a *homogeneous cell process*, which is a special case of self-similar process. Let $\tilde{X}^{(0)} := \exp(\tilde{\xi})$, where $\tilde{\xi}$ is another SNLP with $\tilde{\kappa}$ defined as in (3.1.3), and write $\mathbf{X}^{(0)}$ and $\tilde{\mathbf{X}}^{(0)}$ for two growth-fragmentations associated with $X^{(0)}$ and $\tilde{X}^{(0)}$ respectively (with the same initial size of ancestor $x > 0$), see Section 3.2.3 for their formal construction. The following result partially encompasses Proposition 5 and Corollary 25 in [70].

Theorem 3.1.1 (Homogeneous). *The following statements are equivalent:*

(i) $\kappa = \tilde{\kappa}$*;*

(ii) $X^{(0)}$ *and* $\tilde{X}^{(0)}$ *can be coupled to form a bifurcator (see Definition 3.3.7);*

(iii) the homogeneous growth-fragmentations $\mathbf{X}^{(0)}$ *and* $\tilde{\mathbf{X}}^{(0)}$ *have the same finite-dimensional distributions.*

We hence say that the growth-fragmentation $\mathbf{X}^{(0)}$ is a *homogeneous growth-fragmentation process with characteristic* κ. The function κ serves as *cumulant* for $\mathbf{X}^{(0)}$, in the sense that

$$\mathbf{E}\left[\sum_{x \in \mathbf{X}^{(0)}(t)} x^q\right] = \exp\left(\kappa(q)t\right) \quad \text{for all } q \geq 2 \text{ and } t \geq 0,$$

which is proved in Proposition 3.2.15.

In general, a self-similar cell process with index $\alpha \in \mathbb{R}$ is associated with a Lévy process by Lamperti's representation [59] as follows. Let us define a time-change by

$$\tau_t^{(\alpha)} := \inf\left\{r \geq 0 \ : \ \int_0^r \exp(-\alpha\xi(s))ds \geq t\right\}, \quad t \geq 0,$$

with the convention that $\exp(-\alpha\xi(s)) = 0$ if ξ is killed before s. For every $x > 0$, let us denote by P_x the law of the process

$$X^{(\alpha)}(t) := x \exp(\xi(\tau^{(\alpha)}_{tx^\alpha})), \quad t \geq 0, \tag{3.1.4}$$

with the convention that $X^{(\alpha)}(t) = \partial$ for every $t \geq x^{-\alpha} \int_0^\infty \exp(-\alpha\xi(s))ds$. We know from [59] that for every $c > 0$,

$$\text{the law of } (cX^{(\alpha)}(c^\alpha t), t \geq 0) \text{ under } P_x \text{ is } P_{cx},$$

so we call $X^{(\alpha)}$ a *self-similar cell process with index* α [1]. If $\alpha = 0$, then we simply have $X^{(0)} = x\exp(\xi)$ under P_x, which is indeed a homogeneous cell process.

For $\alpha \neq 0$, we further need to assume that

$$\text{there exists } q > 0 \text{ with } \kappa(q) < 0. \tag{3.1.5}$$

Let $\mathbf{X}^{(\alpha)}$ be a growth-fragmentation associated with $X^{(\alpha)}$ starting from an ancestor cell with initial size $x > 0$. The assumption (3.1.5) ensures the *non-explosion* of the growth-fragmentation $\mathbf{X}^{(\alpha)}$, which means that for every time $t \geq 0$ the multiset $\mathbf{X}^{(\alpha)}(t)$ is locally finite; see [20]. It is also known from a recent work [26] that if $\kappa(q) > 0$ for all $q \geq 0$ and $\alpha \neq 0$, then the growth-fragmentation $\mathbf{X}^{(\alpha)}$ explodes in finite time. Under (3.1.5), it is known from Theorem 2 in [20] that $\mathbf{X}^{(\alpha)}$ keeps the self-similarity: recall that $\mathbf{X}^{(\alpha)}$ starts from an ancestor with initial size x, then for every $c > 0$, the law of $(c\mathbf{X}^{(\alpha)}(c^\alpha t), t \geq 0)$ is the same as a growth-fragmentation associated with $X^{(\alpha)}$ starting from cx. So we call $\mathbf{X}^{(\alpha)}$ *a self-similar growth-fragmentation with index* α.

Let us now present our main result for the self-similar case. Denote $\tilde{X}^{(\tilde{\alpha})}$ for the self-similar cell process of index $\tilde{\alpha} \in \mathbb{R}$ associated with $\tilde{\xi}$ by Lamperti's representation (3.1.4) and let $\tilde{\mathbf{X}}^{(\tilde{\alpha})}$ be the growth-fragmentation driven by $\tilde{X}^{(\tilde{\alpha})}$. Suppose that the respective ancestors of $\tilde{\mathbf{X}}^{(\tilde{\alpha})}$ and $\mathbf{X}^{(\alpha)}$ have the same initial size $x > 0$.

Theorem 3.1.2 (Self-similar). *Suppose that (3.1.5) holds for both κ and $\tilde{\kappa}$, then the following statements are equivalent:*

(i) $\kappa = \tilde{\kappa}$ and $\alpha = \tilde{\alpha}$;

(ii) $X^{(\alpha)}$ and $\tilde{X}^{(\tilde{\alpha})}$ can be coupled to form a bifurcator (see Definition 3.3.7);

(iii) the self-similar growth-fragmentations $\mathbf{X}^{(\alpha)}$ and $\tilde{\mathbf{X}}^{(\tilde{\alpha})}$ have the same finite-dimensional distributions.

[1] The way we define the index of self-similarity α is consistent with the theory of self-similar fragmentations. However, we stress that in the theory of self-similar processes, it is rather $-\alpha$ which is called the index of self-similarity.

Therefore, the law of the self-similar growth-fragmentation $\mathbf{X}^{(\alpha)}$ is characterized by (κ, α). Note that it follows immediately from the self-similarity that if $\tilde{\mathbf{X}}^{(\tilde{\alpha})}$ and $\mathbf{X}^{(\alpha)}$ have the same finite-dimensional distributions, then $\alpha = \tilde{\alpha}$.

Let us outline our proofs. For the homogeneous case, Theorem 3.1.1, we provide a direct proof of the equivalence $(i) \Leftrightarrow (iii)$, by drawing a connection between homogeneous growth-fragmentations and *branching Lévy processes* introduced in [19]. However, this proof cannot be easily extended to the self-similar case. Nevertheless, we can deduce the implication $(iii) \Rightarrow (i)$ in Theorem 3.1.2 from the self-similarity mentioned above and a study of martingales in self-similar growth-fragmentations in [21]. Further, we can construct a bifurcator of $X^{(\alpha)}$ and $\tilde{X}^{(\tilde{\alpha})}$ when $\kappa = \tilde{\kappa}$ and $\alpha = \tilde{\alpha}$ by extending the approach of Pitman and Winkel [70] and using Lamperti's transformation, which gives $(i) \Rightarrow (ii)$. This motivates us to establish the general sufficient condition, Theorem 3.3.9, which is informally stated above. We hence get the implication $(ii) \Rightarrow (iii)$ and complete the proof.

Besides the class of self-similar processes associated with Lévy processes by Lamperti's transformations, the stationary processes driven by Lévy processes, exponential Ornstein-Uhlenbeck type processes (see e.g. [73]), are also natural examples for cell processes. The techniques developed in this paper also open the way to study the growth-fragmentations associated with exponential Ornstein-Uhlenbeck type processes, which will be discussed in Chapter 3.

The rest of this article is organized as follows. We start with working on homogeneous growth-fragmentations in Section 3.2. We first study the bifurcators of homogeneous cell processes, and then characterize the laws of homogeneous growth-fragmentations by using their connections with branching Lévy processes. In Section 3.3, we first provide a non-explosion condition of general Markovian growth-fragmentations, then we introduce the notion of bifurcators for general cell processes and establish our main result, Theorem 3.3.9, a general sufficient condition for two growth-fragmentations to have the same law. Applying Theorem 3.3.9, we complete the proofs of Theorem 3.1.1 and Theorem 3.1.2.

3.2 The homogeneous case

Throughout the rest of this work, we denote by ξ and γ two SNLPs with respective characteristics (σ, c, Λ, k) and $(\sigma_\gamma, c_\gamma, \Lambda_\gamma, k_\gamma)$, and define κ and κ_γ respectively for ξ and γ as in (3.1.3). We also define

$$\bar{z} := \log(1 - \mathrm{e}^z), \quad z \in (-\infty, 0),$$

so that $\mathrm{e}^z + \mathrm{e}^{\bar{z}} = 1$. Note that $z \mapsto \bar{z}$ is an involution, i.e. $\bar{\bar{z}} = z$. For every Lévy measure Λ, we write $\bar{\Lambda}$ for the push-forward measure of Λ via the map $z \mapsto \bar{z}$. We remark that it follows from

(3.1.2) that

$$\Lambda((-\infty,-\log 2])<\infty \quad \text{ and } \quad \bar{\Lambda}([-\log 2,0))<\infty.$$

This section is concerned with growth-fragmentations driven by homogeneous cell processes, and our investigation consists of two parts. We first depict the structure of the family of SNLPs that have the same κ in Section 3.2.1, specifically, we show that they can be derived from each other by the *switching transformations*, which are introduced by Pitman and Winkel [70] to study the bifurcators of fragmenters. We next show that the law of a homogeneous growth-fragmentation associated with $\exp(\xi)$ is characterized by κ. In this direction, we recall the construction of branching Lévy processes introduced by Bertoin [19] in Section 3.2.2 and then build a connection between homogeneous growth-fragmentations and branching Lévy processes in Section 3.2.3. These two results motivate us to extend the conception of bifurcator to general Markov processes and to study the relations between bifurcators and Markovian growth-fragmentations, which will become the object of investigation in Section 3.3.

We will often appeal to the following relation between the SNLPs that have the same κ in terms of their characteristics.

Lemma 3.2.1. *There is $\kappa=\kappa_\gamma$, if and only if*

$$\begin{aligned}
&\Lambda+\bar{\Lambda}=\Lambda_\gamma+\bar{\Lambda}_\gamma, \quad \sigma=\sigma_\gamma, \quad k=k_\gamma \\
&\textit{and} \quad c+\int_{(-\infty,-\log 2)}(1-2\mathrm{e}^z)\Lambda(dz)=c_\gamma+\int_{(-\infty,-\log 2)}(1-2\mathrm{e}^z)\Lambda_\gamma(dz).
\end{aligned}$$

Proof. It is easy to check the *if* part by straightforward calculation. We now prove the *only if* part. If $\kappa=\kappa_\gamma$, then the third order derivatives of $\kappa(q)$ and $\kappa_\gamma(q)$ are equal for every $q>2$, i.e.

$$\int_{(-\infty,0)}\left(\mathrm{e}^{q\bar{z}}\bar{z}^3+\mathrm{e}^{qz}z^3\right)\Lambda(dz)=\int_{(-\infty,0)}\left(\mathrm{e}^{q\bar{z}}\bar{z}^3+\mathrm{e}^{qz}z^3\right)\Lambda_\gamma(dz).$$

Therefore, for every $q>2$ there is

$$\int_{(-\infty,0)}\mathrm{e}^{qz}z^3\left(\Lambda(dz)+\bar{\Lambda}(dz)\right)=\int_{(-\infty,0)}\mathrm{e}^{qz}z^3\left(\Lambda_\gamma(dz)+\bar{\Lambda}_\gamma(dz)\right),$$

which implies that $\Lambda+\bar{\Lambda}=\Lambda_\gamma+\bar{\Lambda}_\gamma$. Iterating this argument over the lower order derivatives of κ and κ_γ, we obtain the other identities in turn. $\square$

3.2.1 Switching transformations and bifurcators

In order to give a construction of bifurcators of homogeneous cell processes, we now generalize the switching transformations between fragmenters in [70] to SNLPs. Let ξ be a SNLP with characteristics (σ,c,Λ,k) and $p:(-\infty,0)\to[0,1]$ be a measurable function, which will serve as switching probability, such that

$$\int_{(-\infty,0)} p(z)\ \Lambda(dz) < \infty. \tag{3.2.1}$$

We shall derive another SNLP $\xi^{[p]}$ from ξ by switching according to p in the following way. At each jump time $t > 0$ of ξ with $z := \Delta\xi(t) = \xi(t) - \xi(t-) < 0$, we *mark* this jump time with success probability $p(z)$ (so with failure probability $1 - p(z)$ we do not mark it), independently of the other jumps. We thus define a point process by the marked jumps:

$$\Delta_1(t) := \begin{cases} \Delta\xi(t) & \text{if } t \text{ is a marked time,} \\ 0 & \text{otherwise.} \end{cases}$$

Implicitly, the killing time ζ is never marked. We stress that the number of marked jump times is locally finite if and only if (3.2.1) holds. Indeed, observing from the property of Lévy processes (see e.g. [12]) that $(\Delta\xi(t), t \geq 0)$ is a Poisson point process with characteristic measure Λ, we have that Δ_1 is a Poisson point process with characteristic measure $\Lambda_1(dz) := p(z)\Lambda(dz)$. Next, we define a point process $\bar{\Delta}_1$ associated with Δ_1 by

$$\bar{\Delta}_1(t) := \begin{cases} \log(1 - \mathrm{e}^{\Delta_1(t)}) & \text{if } \Delta_1(t) \neq 0, \\ 0 & \text{if } \Delta_1(t) = 0. \end{cases}$$

Then $\bar{\Delta}_1$ is a Poisson point process with characteristic measure $\bar{\Lambda}_1(dz) := p(\bar{z})\bar{\Lambda}(dz)$, where $\bar{\Lambda}$ is the image of Λ by the map $z \mapsto \bar{z}$. Therefore, as (3.2.1) holds, the processes

$$\xi_1(t) := \sum_{s \leq t} \Delta_1(s) \quad \text{and} \quad \bar{\xi}_1(t) := \sum_{s \leq t} \bar{\Delta}_1(s)$$

are compound Poisson processes with respective (finite) Lévy measures Λ_1 and $\bar{\Lambda}_1$. We finally define *the switching transform of ξ according to p* by the process

$$\xi^{[p]} := \xi - \xi_1 + \bar{\xi}_1.$$

Lemma 3.2.2. *Let ξ be a SNLP with characteristics (σ, c, Λ, k) and $p : (-\infty, 0) \to [0,1]$ be a measurable function that satisfies (3.2.1). Then the switching transform $\xi^{[p]}$, derived from ξ according to p, is a SNLP with characteristics*

$$\begin{cases} \sigma^{[p]} & := \sigma, \\ \Lambda^{[p]}(dz) & := (1 - p(z))\Lambda(dz) + p(\bar{z})\bar{\Lambda}(dz), \\ c^{[p]} & := c + \int_{(-\infty,0)} (1 - 2\mathrm{e}^{z}) p(z)\Lambda(dz), \\ k^{[p]} & := k, \end{cases} \tag{3.2.2}$$

and Laplace exponent

$$\Phi^{[p]} = \Phi(q) + \int_{(-\infty,0)} \left((1-\mathrm{e}^z)^q - \mathrm{e}^{qz}\right) p(z)\Lambda(dz).$$

Define $\kappa^{[p]}$ *as in* (3.1.3) *for* $\xi^{[p]}$, *then* $\kappa^{[p]} = \kappa$. *Further,*

$$\tau := \inf\left\{t \geq 0 \ : \ \xi(t) \neq \xi^{[p]}(t)\right\}$$

has an exponential distribution with parameter $\int_{(-\infty,0)\setminus\{-\log 2\}} p(z) \ \Lambda(dz) < \infty$. *Moreover, if* $\tau < \infty$ *then* τ *is a jump time of both* ξ *and* $\xi^{[p]}$ *with*

$$\exp(\xi(\tau)) + \exp(\xi^{[p]}(\tau)) = \exp(\xi(\tau-)).$$

Proof. The Lévy processes $(\xi - \xi_1)$ and ξ_1 are independent since they never jump at the same time. For the same reason, the Lévy processes $(\xi - \xi_1)$ and $\bar{\xi}_1$ are also independent. Therefore, the Laplace exponent of $\xi^{[p]}$ is $\Phi - \Phi_1 + \bar{\Phi}_1$, where Φ_1 and $\bar{\Phi}_1$ are respective Laplace exponents of ξ_1 and $\bar{\xi}_1$. So we get (3.2.2) and thus check that $\kappa^{[p]} = \kappa$ by straightforward calculation.

We next observe from the construction of $\xi^{[p]}$ that

$$\inf\left\{t \geq 0 \ : \ \xi(t) \neq \xi^{[p]}(t)\right\} = \inf\left\{t \geq 0 \ : \ \Delta_1(t) \neq 0 \text{ and } \Delta_1(t) \neq -\log 2\right\},$$

which implies the second part of the statement. $\square$

Remark 3.2.3. *It follows from* (3.1.2) *that for every* $a \geq 2$ *the function* $z \mapsto (1-\mathrm{e}^z)^a$ *satisfies* (3.2.1). *However, the function* $z \mapsto (1-\mathrm{e}^z)$, *which would correspond to the size-biased pick between* $\exp(\Delta\xi(t))$ *and* $(1-\exp(\Delta\xi(t)))$ *(see Section 2.2 in [70]) cannot satisfy* (3.2.1) *unless* $\int_{(-\infty,0)}(|z| \wedge 1)\Lambda(dz) < \infty$.

Lemma 3.2.4. *If* $\kappa_\gamma = \kappa$, *then for every measurable function* $p\colon (-\infty,0) \to [0,1]$ *such that*

$$\int_{(-\infty,0)} p(z)\Lambda(dz) < \infty \quad \text{and} \quad p(z) + p(\bar{z}) = 1 \text{ for every } z \in (-\infty,0), \tag{3.2.3}$$

there is $\int_{(-\infty,0)} p(z)\Lambda_\gamma(dz) < \infty$ *and* $\gamma^{[p]} \overset{d}{=} \xi^{[p]}$.

The function $z \mapsto \mathbf{1}_{\{z<-\log 2\}} + \frac{1}{2}\mathbf{1}_{\{z=-\log 2\}}$ gives an example that satisfies (3.2.3).

Proof of Lemma 3.2.4. As $\kappa_\gamma = \kappa$, it follows from Lemma 3.2.1 and (3.1.2) that $\Lambda_\gamma - \Lambda$ is a finite signed measure and hence we have

$$\int_{(-\infty,0)} p(z)\Lambda_\gamma(dz) \leq \int_{(-\infty,0)} p(z)\Lambda(dz) + \int_{(-\infty,0)} |\Lambda_\gamma - \Lambda|(dz) < \infty.$$

So the switching transforms $\gamma^{[p]}$ and $\xi^{[p]}$ are well-defined. As (3.2.3) holds, by combining Lemma 3.2.1 and Lemma 3.2.2, we get that the characteristics of $\gamma^{[p]}$ are the same as those of $\xi^{[p]}$. $\square$

We next see that the SNLPs that have the same κ are related to each other via the switching transformations.

Proposition 3.2.5. *If $\kappa_\gamma = \kappa$, then $\gamma \overset{d}{=} \xi^{[p]}$, where p is the measurable function defined by Radon-Nikodym derivative*

$$p(z) := \bar{\Lambda}_\gamma(dz)/(\Lambda_\gamma(dz) + \bar{\Lambda}_\gamma(dz)).$$

Proof. Observe that

$$\int_{(-\infty,0)} p(z)\Lambda_\gamma(dz) \le \int_{(-\infty,-\log 2)} \Lambda_\gamma(dz) + \int_{(-\log 2,0)} \bar{\Lambda}_\gamma(dz) < \infty,$$

then the switching transform $\gamma^{[p]}$ is well-defined, and we deduce from Lemma 3.2.2 that $\gamma^{[p]} \overset{d}{=} \gamma$. Note that $p(z) + p(\bar{z}) = 1$ for every $z \in (-\infty, 0)$, then it follows from Lemma 3.2.4 that $\xi^{[p]}$ is also well-defined and $\xi^{[p]} \overset{d}{=} \gamma^{[p]}$. So we conclude that $\gamma \overset{d}{=} \xi^{[p]}$. $\square$

We finally present a construction of a bifurcator of homogeneous cell processes, which has the following precise definition.

Definition 3.2.6. *A pair of homogeneous cell processes (X, Y) is a* **bifurcator** *if it satisfies the following properties:*

(i) Let $\tau := \inf\{t \ge 0 : X(t) \ne Y(t)\}$. There is almost surely either $\tau = \infty$ or the identity

$$X(\tau) + Y(\tau) = X(\tau-) = Y(\tau-).$$

(ii) (Asymmetric Markov branching property) Conditionally given $\tau > t$, the pair $(X(r + t)/X(t), Y(r+t)/Y(t))_{r\ge 0}$ is a copy of (X, Y); conditionally given $\tau \le t$, the two processes $(X(r+t)/X(t))_{r\ge 0}$ and $(Y(r+t)/Y(t))_{r\ge 0}$ are independent copies of X and Y respectively.

This definition generalizes bifurcators of fragmenters in [70]. We shall later extend this notion to general cell processes, see Definition 3.3.7.

Lemma 3.2.7. *If $\kappa = \kappa_\gamma$, then there exists a bifurcator of homogeneous cell processes (X, Y), such that the marginal laws of X and Y are the laws of $\exp(\xi)$ and $\exp(\gamma)$ respectively.*

Proof. Since $\kappa = \kappa_\gamma$, we can build as in Proposition 3.2.5 the switching transform $\xi^{[p]}$ derived from ξ such that $\xi^{[p]} \overset{d}{=} \gamma$. We stress that ξ and $\xi^{[p]}$ are still coupled after the switching time $\tau := \inf\{t \ge 0 : \xi(t) \ne \xi^{[p]}(t)\}$. However, let us define a process Y by

$$Y(t) := \mathbf{1}_{\{t<\tau\}} \exp(\xi(t)) + \mathbf{1}_{\{t\ge\tau\}} \exp(\xi^{[p]}(\tau) + \gamma'(t - \tau)), \quad t \ge 0,$$

where γ' is a copy of γ, independent of $\xi^{[p]}$ and ξ. Then we easily check that $Y \overset{d}{=} \exp(\gamma)$ and the pair of homogeneous cell processes $(X := \exp(\xi), Y)$ satisfies Definition 3.2.6. $\square$

3.2.2 Binary branching Lévy processes

Let ξ_b be a SNLP with characteristics $(\sigma_b, c_b, \Lambda_b, k_b)$ and Π_b be a Lévy measure on $[-\log 2, 0)$ that satisfies

$$\int_{[-\log 2,0)} (1 \wedge z^2)\Pi_b(dz) < \infty. \tag{3.2.4}$$

Informally speaking, a *binary branching Lévy process (BBLP)* introduced in [19] models the evolution of a particle system, in which each particle moves in $\mathbb{R}$ according to the SNLP ξ_b, independently of the other particles, and at rate $\Pi_b(dz)$ each particle gives birth to two children scattered on $\mathbb{R}$, whose initial positions relative to the position of the parent at death are given by z and $\bar{z} = \log(1-\mathrm{e}^z)$. We further add a properly chosen positive drift for the entire system, which is an analogue of the compensation term in the Lévy-Khintchine formula (3.1.1), so that the particles in this system do not all shift to $-\infty$ instantaneously. Proposition 3 in [20] establishes a close connection between BBLPs and homogeneous growth-fragmentations. We will extend this connection in the next subsection. Before that, we recall some basic facts of BBLPs in this subsection.

Let us represent the formal construction of BBLPs in [19], starting with the case when the branching occurs with a finite intensity, i.e. $\Pi_b([-\log 2, 0)) < \infty$. Write $\mathcal{U}_2 := \bigcup_{n=0}^{\infty}\{\ell, r\}^n$ for the binary Ulam-Harris tree with $\{\ell, r\}^0 := \emptyset$ by convention, so for every $i \in \mathbb{N}$, an element in $\{\ell, r\}^i$ is a word $v = (n_1, n_2, \ldots, n_i)$ composed of i letters of the alphabet $\{\ell, r\}$. We write $|v| := i$ for the generation of v and $(v\ell, vr)$ for its children, where $v\ell$ would be referred to as the *left* child and vr as the *right* child. For every $j \leq |v|$, we denote by $[v]_j := (n_1, n_2, \ldots, n_j)$ the ancestor of v at the j-th generation.

Definition 3.2.8. *Let ξ_b be a SNLP with characteristics $(\sigma_b, c_b, \Lambda_b, k_b)$ and Π_b be a finite measure on $[-\log 2, 0)$. We consider three independent processes $(\lambda_v)_{v\in\mathcal{U}_2}$, $(L_v)_{v\in\mathcal{U}_2}$ and $(D_v)_{v\in\mathcal{U}_2}$ such that:*

- *$(\lambda_v)_{v\in\mathcal{U}_2}$ is a family of i.i.d. exponential variables with parameter $\Pi_b([-\log 2, 0))$.*
- *$(L_v)_{v\in\mathcal{U}_2}$ is a family of independent SNLP distributed as*

$$\xi_b(t) + \left(\int_{[-\log 2,0)} (1 - \mathrm{e}^z)\Pi_b(dz)\right)t, \quad t \geq 0.$$

- *$(D_{v\ell}, D_{vr})_{v\in\mathcal{U}_2}$ is a family of i.i.d. random variables, such that $D_{v\ell}$ is distributed according to the conditional probability $\Pi_b(\cdot \mid [-\log 2, 0))$ and $D_{vr} = \overline{D_{v\ell}} = \log(1 - \exp(D_{v\ell})) \leq D_{v\ell}$.*

Define for every $v \in \mathcal{U}_2$ the birth time by $\beta_v := \sum_{j=0}^{|v|-1} \lambda_{[v]_j}$, and iteratively the positions of its children at birth by $(a_{vi} = a_v + L_v(\lambda_v) + D_{vi}, i \in \{\ell, r\})$, with $a_\emptyset = 0$. We agree that $L_v(s) = -\infty$ if L_v is killed before s. Then the positions of the particles alive at time $t \geq 0$ form a multiset of elements in $\mathbb{R}$:

$$\mathbf{Z}(t) := \{\!\{ a_v + L_v(t - b_v) \ : \ v \in \mathcal{U}_2, \beta_v \leq t < \beta_v + \lambda_v \}\!\}.$$

The process $(\mathbf{Z}(t), t \geq 0)$ is a **binary branching Lévy process (BBLP) with characteristics** $(\sigma_b, c_b, \Lambda_b, k_b, \Pi_b)$.

Remark 3.2.9. *A multiset $\mathcal{I}$ could be equivalently viewed as the point measure $\sum_{i \in \mathcal{I}} \delta_i$, where δ stands for the Dirac mass. So we can identify $\mathbf{Z}$ with a point process.*

We next extend the construction to infinite branching intensity and suppose that

$$\Pi_b([-\log 2, 0)) = \infty.$$

For every $\mathtt{d} \leq -\log 2$, let us set

$$\Pi_b^{\{\mathtt{d}\}} := \mathbf{1}_{\{[-\log 2, \bar{\mathtt{d}})\}} \Pi_b, \quad \Lambda_b^{\{\mathtt{d}\}} := \Lambda_b + \mathbf{1}_{\{[\bar{\mathtt{d}}, 0)\}} \Pi_b. \tag{3.2.5}$$

We know from Lemma 3 in [19] that we can construct a family of processes $(\mathbf{Z}^{\mathtt{d}}, -\infty < \mathtt{d} \leq -\log 2)$ in the same probability space, with each $\mathbf{Z}^{\mathtt{d}}$ a BBLP with characteristics $(\sigma_b, c_b, \Lambda_b^{\{\mathtt{d}\}}, k_b, \Pi_b^{\{\mathtt{d}\}})$ in the sense of Definition 3.2.8 (we stress that (3.2.4) assures that $\Pi_b^{\{\mathtt{d}\}}$ is a finite measure), such that for every $\mathtt{d} \leq \mathtt{d}' \leq -\log 2$ there is $(\mathbf{Z}^{\mathtt{d}})^{\{\mathtt{d}'\}} = \mathbf{Z}^{\mathtt{d}'}$, where $(\mathbf{Z}^{\mathtt{d}})^{\{\mathtt{d}'\}}$ is the system derived from $\mathbf{Z}^{\mathtt{d}}$ by keeping at each branching event the child particle that is closer to the mother, and suppressing the other child particle (together with its offspring) whenever it is born at distance from its mother $\geq |\mathtt{d}'|$.

Definition 3.2.10. *In the notation above, suppose that Π_b is a Lévy measure on $[-\log 2, 0)$ that verifies (3.2.4). Then the limit process (by monotonicity in the sense of multiset inclusion)*

$$\mathbf{Z}(t) := \lim_{\mathtt{d} \to -\infty} \uparrow \mathbf{Z}^{\mathtt{d}}(t), \quad t \geq 0$$

is a **BBLP with characteristics** $(\sigma_b, c_b, \Lambda_b, k_b, \Pi_b)$.

Remark 3.2.11. *Our notation is slightly different from that of [19]. In the sense of Definition 2 in [19], a BBLP with characteristics $(\sigma_b, c_b, \Lambda_b, k_b, \Pi_b)$ is characterized by $(\sigma_b, c_b - k_b, \mu_b)$, where μ_b is a measure on the space*

$$\Big\{ (r_1, r_2, \ldots) \ : \ r_i \in [-\infty, 0] \text{ for } i \in \mathbb{N}, \ \mathrm{e}^{r_1} + \mathrm{e}^{r_2} + \ldots \leq 1, \ \text{and } r_1 \geq r_2 \geq \ldots \Big\},$$

and is given by the sum of the following three measures: the image of Λ_b by the map $z \mapsto (z, -\infty, -\infty, \ldots)$, the image of Π_b by the map $z \mapsto (z, \bar{z}, -\infty, -\infty, \ldots)$ and $k_b \delta_{(-\infty, -\infty, \ldots)}$.

Let Φ_b be the Laplace exponent of the SNLP ξ_b with characteristics $(\sigma_b, c_b, \Lambda_b, k_b)$. Introduce $\kappa_b : [0,\infty) \to (-\infty,\infty]$ by

$$\kappa_b(q) := \Phi_b(q) + \int_{[-\log 2,0)} \left(\mathrm{e}^{qz} + (1-\mathrm{e}^z)^q - 1 + q(1-\mathrm{e}^z)\right) \Pi_b(dz), \quad q \geq 0,$$

then κ_b serves as cumulant for the BBLP $\mathbf{Z}$. Specifically, we know from Theorem 1 in [19] that for every $q \geq 2$, there is $\kappa_b(q) < \infty$ and

$$\mathbb{E}\left[\sum_{z\in\mathbf{Z}(t)} \mathrm{e}^{qz}\right] = \mathrm{e}^{\kappa_b(q)t} \quad \text{for all } t \geq 0. \tag{3.2.6}$$

We now check that if $\Lambda_b = 0$, then the cumulant determines the distribution of the BBLP in the following sense.

Lemma 3.2.12. *Let* $\mathbf{Z}$ *and* $\mathbf{Z}'$ *be two BBLPs with respective characteristics* $(\sigma_b, c_b, \Lambda_b, k_b, \Pi_b)$ *and* $(\sigma_b', c_b', \Lambda_b', k_b', \Pi_b')$. *If* $\Lambda_b = \Lambda_b' = 0$ *and their cumulants* $\kappa_b = \kappa_b'$, *then* $\mathbf{Z}$ *and* $\mathbf{Z}'$ *have the same law.*

Proof. Since the third order derivatives of κ_b' and κ_b are equal for all $q > 2$, by a similar argument as in the proof of Lemma 3.2.1, we find that $\Pi_b' + \bar{\Pi}_b' = \Pi_b + \bar{\Pi}_b$. As Π_b' and Π_b are supported on $[\log 2, 0]$, we hence find that $\Pi_b' = \Pi_b$. By iterating this argument over the lower order of derivatives, we conclude that $\mathbf{Z}$ and $\mathbf{Z}'$ have the same characteristics, thus the same law. $\square$

3.2.3 Homogeneous growth-fragmentations

For every $x > 0$, write P_x for the law of the homogeneous cell process $X := x\exp(\xi)$, where ξ is a SNLP with κ defined as in (3.1.3). If ξ is killed at a time ζ, then by convention we denote $X(t) = \partial$ for all $t \geq \zeta$, where ∂ is the cemetery state. Let $\mathbf{X}$ be a homogeneous growth-fragmentation associated with X, which was informally described in the Introduction. By connecting to branching Lévy processes, we shall prove in this section that the law of $\mathbf{X}$ is characterized by the cumulant function κ.

In that direction, let us present the rigorous construction of $\mathbf{X}$, which is only a slight modification of that in [20]. We start with listing the jumps of X in the following way. Fix $q > 2$ and $K > \kappa(q)$. Recalling that the jump process $\Delta\xi$ is a Poisson point process with characteristic measure Λ and using the compensation formula (see e.g. [12]), we get for every $x > 0$

$$\begin{aligned} E_x\left[\sum_{0\leq s} |\Delta X(s)|^q \mathrm{e}^{-Ks}\right] &= E_x\left[\sum_{0\leq s} X(s-)^q (1-\mathrm{e}^{\Delta\xi(s)})^q \mathrm{e}^{-Ks}\right] \\ &= E_x\left[\int_0^\infty \mathrm{e}^{-Ks} X(s-)^q ds \int_{(-\infty,0)} (1-\mathrm{e}^z)^q \Lambda(dz)\right], \end{aligned}$$

where E_x stands for mathematical expectation under P_x. Using the definition of Φ and κ, we deduce that

$$E_x\left[\sum_{0\leq s}|\Delta X(s)|^q\mathrm{e}^{-Ks}\right] = \frac{\kappa(q)-\Phi(q)}{K-\Phi(q)}x^q,$$

which implies that P_x-almost surely

$$\sum_{s\geq 0}|\Delta X(s)|^q\mathrm{e}^{-Ks}<\infty.$$

We may therefore list the jump times of X in a sequence $(t_i, i\in\mathbb{N})$ such that $(|\Delta X(t_i)|^q\mathrm{e}^{-Kt_i}, i\in\mathbb{N}))$ is decreasing. By convention, if X has a finite number of jumps, then the tail of this sequence is filled with ∞ with $\Delta X(\infty)=\partial$. In the sequel, **the i-th jump time of X** shall always refer to the i-th element t_i in this sequence.

Let us give some basic notations. Let $\mathcal{U}:=\bigcup_{i=0}^{\infty}\mathbb{N}^i$ be the Ulam-Harris tree, by convention $\mathbb{N}^0=\{\emptyset\}$. An element $u\in\mathcal{U}$ is a finite sequence of natural numbers $u=(n_1,\dots,n_{|u|})$ where $|u|\in\mathbb{N}$ stands for the generation of u. We write $u_-=(n_1,\dots,n_{|u|-1})$ for its mother and $uk=(n_1,\dots n_{|u|},k)$ for its k-th daughter with $k\in\mathbb{N}$. We also denote $[u]_i=(n_1,\dots,n_i)$ for every $i\leq|u|$ with $[u]_0=\emptyset$ by convention.

We next construct the **cell system driven by** X, which is a family of homogeneous cell processes indexed by $\mathcal{U}$

$$\mathcal{X}:=(\mathcal{X}_u,\ u\in\mathcal{U}),$$

where each $\mathcal{X}_u$ depicts the evolution of the size of the cell indexed by u as time passes. Specifically, we fix an arbitrary $x>0$, which is the initial size of the ancestor cell. Then we set the birth time of $\emptyset$ at $b_\emptyset:=0$ and let the life career $\mathcal{X}_\emptyset=(\mathcal{X}_\emptyset(t),t\geq 0)$ be a process of law P_x. Given the life path of $\mathcal{X}_\emptyset$, then we generate the first generation. For $i\in\mathbb{N}$, say the i-th jump time of $\mathcal{X}_\emptyset$ is t_i and $x_i:=-\Delta\mathcal{X}_\emptyset(t_i)$, we then set $b_i=t_i$ and build a sequence of conditionally independent processes $(\mathcal{X}_i)_{i\in\mathbb{N}}$ with respective conditional distribution P_{x_i}. By convention, if $t_i=\infty$ (which means that $\mathcal{X}_\emptyset$ has less than i jumps), then we agree that the cell i as well as all its progeny have degenerate life careers, i.e. for every $v\in\mathcal{U}$ we set $\mathcal{X}_{iv}\equiv\partial$ and $b_{iv}=\infty$. We continue in this way to construct higher generations recursively. Write $\mathcal{P}_x$ for the law of this cell system $\mathcal{X}$ (recall that $x>0$ indicates the initial size of the Eve $\emptyset$, i.e. $\mathcal{X}_\emptyset(0)=x$). According to [50], the probability distribution $\mathcal{P}_x$ indeed exists and is uniquely determined by the above description.

Finally, for every $t\geq 0$ let $\mathbf{X}(t)$ be the multiset whose elements are sizes of the cells alive at time t, i.e.

$$\mathbf{X}(t):=\{\!\!\{\mathcal{X}_u(t-b_u)\ :\ u\in\mathcal{U}, b_u\leq t\}\!\!\},$$

then we refer to $\mathbf{X}=(\mathbf{X}(t),t\geq 0)$ as a **growth-fragmentation process driven by** X and write $\mathbf{P}_x$ for the law of $\mathbf{X}$ under $\mathcal{P}_x$.

Remark 3.2.13. *The construction of the cell system $\mathcal{X}$ is only a slight modification of that of a cell system in [20], and that of a general branching process (also called Crump-Mode-Jagers process) in [50]. The only difference lies in the fact that, in [20] daughters are listed in decreasing order of the sizes at birth, and in [50] daughters are enumerated by their birth times. However, in full generality, it is not always possible to enumerate the jumps of a homogeneous process X in decreasing order of jump sizes or increasing order of jump times.*

Remark 3.2.14. *If we use a different way to enumerate the jumps of X, it is intuitively clear that the new cell system is the same as the original one, up to a permutation of $\mathcal{U}$. Thus the growth-fragmentation $\mathbf{X}$ obviously does not depend on the method of enumeration and the law of $\mathbf{X}$ is determined by X.*

We now present a connection between homogeneous growth-fragmentation processes and BBLPs.

Proposition 3.2.15. *Let ξ be a SNLP with characteristics (σ, c, Λ, k) and κ defined as in (3.1.3) and $\mathbf{X}$ be a homogeneous growth-fragmentation process (starting from 1) driven by $X := \exp(\xi)$. Then the process $\log \mathbf{X}$ is the unique (in law) BBLP with cumulant κ and $\Lambda_b = 0$. Specifically, $\log \mathbf{X}$ has characteristics $(\sigma, c_b, 0, k, \Pi_b)$, where*

$$c_b = c + \int_{(-\infty,-\log 2)} \left(1 - 2\mathrm{e}^z\right)\Lambda(dz), \ \ \textit{and} \ \ \ \Pi_b = \mathbf{1}_{\{(-\log 2,0)\}}(\Lambda + \bar{\Lambda}) + \frac{1}{2}\mathbf{1}_{\{-\log 2\}}(\Lambda + \bar{\Lambda}).$$

In particular, we have

$$\mathbf{E}\left[\sum_{x\in\mathbf{X}(t)} x^q\right] = \exp\left(\kappa(q)t\right) \quad \textit{for all } q \geq 2 \textit{ and } t \geq 0.$$

Remark 3.2.16. *Write ν for the image of Λ by $z \mapsto (\max(\mathrm{e}^z, 1 - \mathrm{e}^z), \min(\mathrm{e}^z, 1 - \mathrm{e}^z))$ and $\mathbf{0} := (0, 0, \dots) \in \mathcal{S}$. Then the homogeneous growth-fragmentation $\mathbf{X}$ is a compensated fragmentation process with characteristics $\left(\sigma, c - k + \int_{(-\infty,-\log 2)}(1 - 2\mathrm{e}^z)\Lambda(dz), \nu + k\delta_{\mathbf{0}}\right)$ in the sense of [19]. Loosely speaking, a compensated fragmentation is the limit of properly dilated homogeneous fragmentations, see Corollary 4 in [19].*

Proposition 3.2.15 extends Corollary 25 in [70] (which treats the case when ξ is the negative of a pure-jump subordinator) and Proposition 3 in [20] (for the case when the Lévy measure Λ of ξ satisfies $\Lambda\big((-\infty, -\log 2)\big) = 0$). Before tackling the proof of Proposition 3.2.15, let us provide a variation of Theorem 3.1.1, which summarizes the discussion in this section.

Corollary 3.2.17. *Let ξ and $\tilde{\xi}$ be two SNLPs with respective cumulants κ and $\tilde{\kappa}$ defined as in (3.1.3). Let $\mathbf{X}$ and $\tilde{\mathbf{X}}$ be the homogeneous growth-fragmentations associated with ξ and $\tilde{X}$ respectively (with the same initial size of ancestor $x > 0$), The following statements are equivalent:*

(i) $\kappa = \tilde{\kappa}$;

(ii) $\tilde{\xi}$ *has the same law as a switching transform of* ξ*;*

(iii) the homogeneous growth-fragmentations $\mathbf{X}$ *and* $\tilde{\mathbf{X}}$ *have the same finite-dimensional distributions.*

Proof of Corollary 3.2.17. $(i) \Leftrightarrow (ii)$: The two directions follow respectively from Proposition 3.2.5 and Lemma 3.2.2.

$(i) \Leftrightarrow (iii)$: We know from Proposition 3.2.15 that $\log \mathbf{X}$ and $\log \tilde{\mathbf{X}}$ are BBLPs with respective cumulants κ and $\tilde{\kappa}$. If $\mathbf{X}$ and $\tilde{\mathbf{X}}$ have the same finite-dimensional distributions, then so do the BBLPs $\log \mathbf{X}$ and $\log \tilde{\mathbf{X}}$, and in particular their cumulants are the same. Conversely, if $\kappa = \tilde{\kappa}$, then we deduce from Lemma 3.2.1 or Lemma 3.2.12 that the BBLPs $\log \mathbf{X}$ and $\log \tilde{\mathbf{X}}$ have the same characteristics, thus the same finite-dimensional distributions. $\square$

The rest of this section is devoted to the proof of Proposition 3.2.15.

Proof of Proposition 3.2.15. The idea of the proof is similar to that of Proposition 3 in [20]. Let $\mathbf{Z}$ be a BBLP with characteristics $(\sigma, c_b, 0, k, \Pi_b)$ and write $(\mathbf{Z}^{\mathtt{d}}, -\infty < \mathtt{d} \leq -\log 2)$ for the family of BBLPs as in Definition 3.2.8, each $\mathbf{Z}^{\mathtt{d}}$ a BBLP with characteristics $(\sigma, c_b, \mathbf{1}_{\{[\bar{\mathtt{d}},0)\}}\Pi_b, k, \mathbf{1}_{\{[-\log 2,\bar{\mathtt{d}})\}}\Pi_b)$, such that

$$\mathbf{Z}(t) = \lim_{\mathtt{d}\to-\infty} \uparrow \mathbf{Z}^{\mathtt{d}}(t), \quad t \geq 0.$$

We shall check for every $\mathtt{d} \in (-\infty, -\log 2)$ that $\exp(\mathbf{Z}^{\mathtt{d}})$ has the same dynamics as a truncated cell system associated with the cell process $X = \exp(\xi)$, in which each cell $u \in \mathcal{U}$ is killed at the first instant s with $\mathcal{X}_u(s) \leq \mathrm{e}^{\mathtt{d}}\mathcal{X}_u(s-)$, together with her future descendants (born at time $> s$); furthermore, for each $j \in \mathbb{N}$ the daughter cell uj is killed at birth (together with its descendants) whenever her size is less than or equal to $\mathrm{e}^{\mathtt{d}}$ times the size of her mother immediately before the birth event, i.e. $\mathcal{X}_{uj}(0) \leq \mathrm{e}^{\mathtt{d}}\mathcal{X}_u(b_{uj}-)$. Letting $\mathtt{d} \to -\infty$, we conclude from Definition 3.2.8 and the monotonicity that $\log \mathbf{X}$ has the same distribution as $\mathbf{Z}$. Then it is straightforward to check that $\log \mathbf{X}$ indeed has cumulant κ and the identity in the proposition thus follows from (3.2.6). The uniqueness of $\log \mathbf{X}$ follows from Lemma 3.2.12.

So it remains to prove that $\exp(\mathbf{Z}^{\mathtt{d}})$ indeed has the same law as the truncated cell system. In this direction, let us construct an auxiliary particle system as follows, which is a minor modification of Definition 3.2.8. Fix an arbitrary $\mathtt{d} < -\log 2$. Let us consider three independent sequences of processes $(\lambda_v)_{v\in\mathcal{U}_2}$, $(L_v)_{v\in\mathcal{U}_2}$ and $(D_{v\ell}, D_{vr})_{v\in\mathcal{U}_2}$ such that:

- $(\lambda_v)_{v\in\mathcal{U}_2}$ is a family of i.i.d. exponential variables with parameter $\Lambda((\infty, \bar{\mathtt{d}}))$;
- $(L_v)_{v\in\mathcal{U}_2}$ is a family of independent copies of SNLP $\tilde{\xi}$ with characteristics $(\sigma, \tilde{c}, \mathbf{1}_{\{[\bar{\mathtt{d}},0)\}}\Lambda, k)$ where $\tilde{c} := c + \int_{(-\infty,\bar{\mathtt{d}})}(1 - \mathrm{e}^z)\Lambda(dz)$.

- $(D_{v\ell}, D_{vr})_{v\in\mathcal{U}_2}$ is a family of i.i.d. pairs of random variables such that each $D_{v\ell}$ is distributed according to the conditional probability $\Lambda(\cdot \mid (-\infty, \bar{\mathtt{d}}))$ and $D_{vr} = \overline{D_{v\ell}} = \log(1-\exp(D_{vr}))$.

Write $\beta_v := \sum_{j=0}^{|v|-1} \lambda_{[v]_j}$ for the birth time, and define by induction $a_{vi} = a_v + L_v(\lambda_v) + D_{vi}$ for $i \in \{\ell, r\}$ with $a_\emptyset = 0$. So we define $\mathbf{L}$ by

$$\mathbf{L}(t) := \{\!\{a_v + L_v(t-\beta_v) \ : \ v \in \mathcal{U}_2, \beta_v \le t < \beta_v + \lambda_v\}\!\}, \quad t \ge 0.$$

We stress that unlike in Definition 3.2.8, $\mathbf{1}_{\{(-\infty,\bar{\mathtt{d}})\}}\Lambda$ is not supported on $[-\log 2, 0)$, so $D_{v\ell}$ may be possibly smaller than D_{vr}. However, we may obtain a BBLP by changing the indices of the particles. Specifically, let us define a bijection $h: \mathcal{U}_2 \to \mathcal{U}_2$ in the following way. Let $h(\emptyset) := \emptyset$. Given $h(v)$ with $v \in \mathcal{U}_2$ by induction, then we assign the index of $\max(D_{h(v)\ell}, D_{h(v)r})$ to $h(v\ell)$ and let $h(vr)$ be the sister of $h(v\ell)$. We therefore define $(D'_{v\ell}, D'_{vr}) := (D_{h(v\ell)}, D_{h(vr)})$, $\beta'_v := \beta_{h(v)}$ and $L'_v := L_{h(v)}$ for each $v \in \mathcal{U}_2$, and further define recursively $a'_{vi} := a'_v + L'_v(\lambda'_v) + D'_{vi}$. As h is a bijection, it is plain that

$$\mathbf{L}(t) = \{\!\{a'_v + L'_v(t-\beta'_v) \ : \ v \in \mathcal{U}_2, \beta'_v \le t < \beta'_v + \lambda'_v\}\!\}, \quad t \ge 0.$$

Let

$$\Pi_{\mathbf{L}} = \frac{1}{2}\mathbf{1}_{\{-\log 2\}}(\Lambda + \bar{\Lambda}) + \mathbf{1}_{\{(-\log 2, \bar{\mathtt{d}})\}}\Lambda + \mathbf{1}_{\{(-\log 2, 0)\}}\bar{\Lambda},$$

then $\Pi_{\mathbf{L}}$ is supported on $[-\log 2, 0)$ and we observe that $((D'_{h(v)\ell}, D'_{h(v)r}), v \in \mathcal{U}_2)$ is a family of i.i.d. random variables such that $D'_{h(v)\ell}$ has conditional law $\Pi_{\mathbf{L}}(\cdot \mid [-\log 2, 0))$ and $D'_{h(v)r} = \overline{D'_{h(v)\ell}}$, that $(\beta'_v, v \in \mathcal{U}_2)$ is a family of i.i.d. exponential variables with parameter $\Lambda((-\infty, \bar{\mathtt{d}})) = \Pi_{\mathbf{L}}([-\log 2, 0))$ and that $(L'_v, v \in \mathcal{U}_2)$ is a family of independent copies of $\tilde{\xi}$. Using this point of view, we hence deduce that $\mathbf{L}$ is a BBLP as in Definition 3.2.8, with characteristics $(\sigma, c_b, \Lambda_{\mathbf{L}} := \mathbf{1}_{\{[\bar{\mathtt{d}},0)\}}\Lambda, k, \Pi_{\mathbf{L}})$, where we have used the fact that

$$\tilde{c} - \int_{[-\log 2, 0)} (1-\mathrm{e}^z)\Pi_{\mathbf{L}}(dz) = c + \int_{(-\infty, -\log 2)} (1 - 2\mathrm{e}^z)\Lambda(dz) = c_b.$$

Let us next give some remarks on the *left-most* branch of the particle system $\mathbf{L}$, that is the process obtained by concatenating the segments of size processes of particles $\{\emptyset, \ell, \ell\ell, \ell\ell\ell, \dots\} := \ell^{\mathbb{N}\cup\{0\}} := \ell^{\mathbb{N}} \subset \mathcal{U}_2$:

$$A_{\ell^{\mathbb{N}}}(t) := \sum_{v \in \ell^{\mathbb{N}}} \mathbf{1}_{\{\beta_v \le t < \beta_v + \lambda_v\}}(a_v + L_v(t - \beta_v)), \quad t \ge 0. \tag{3.2.7}$$

Using elementary properties of Lévy processes, we find that $A_{\ell^{\mathbb{N}}}$ has the same distribution as ξ. We also notice that for every time $t \ge 0$ when $\Delta A_{\ell^{\mathbb{N}}}(t) < \bar{\mathtt{d}}$, that is equivalently $\exp(A_{\ell^{\mathbb{N}}})$ has a jump of size $-\Delta \exp(A_{\ell^{\mathbb{N}}}(t)) > \mathrm{e}^{\mathtt{d}} \exp(A_{\ell^{\mathbb{N}}}(t-))$, there is $t = \beta_v + \lambda_v$ for a certain $v \in \ell^{\mathbb{N}}$.

A fortiori, for every $s \geq 0$ such that $\Delta A_{\ell^{\mathbb{N}}}(s) < \mathtt{d} < \bar{\mathtt{d}}$, that is equivalently $\exp(A_{\ell^{\mathbb{N}}}(s)) \leq \mathrm{e}^{\mathtt{d}} \exp(A_{\ell^{\mathbb{N}}}(s-))$, there is $s = \beta_w + \lambda_w$ for a certain $w \in \ell^{\mathbb{N}}$.

We finally consider the process $\hat{\mathbf{L}}$, which is associated with a system derived from $\mathbf{L}$, by suppressing for each $v \in \mathcal{U}_2$ the child that corresponds to $D_{v\ell}$ whenever $D_{v\ell} \leq \mathtt{d}$. So we can explain the dynamics of $\exp(\hat{\mathbf{L}})$ as follows. This system starts with an Eve cell whose size evolves according to $\mathcal{X}_\emptyset := \exp(A_{\ell^{\mathbb{N}}})$, and the Eve cell is killed (together with her future descendance) at the first instant $s \geq 0$ when there is $\mathcal{X}_\emptyset(s) \leq \mathrm{e}^{\mathtt{d}} \mathcal{X}_\emptyset(s-)$. Further, for each time $t \leq s$ when $\mathcal{X}_\emptyset$ has a jump of size $y := -\Delta \mathcal{X}_\emptyset(t) > \mathrm{e}^{\mathtt{d}} \mathcal{X}_\emptyset(t-)$, there is $t = \beta_v + \lambda_v$ for a certain $v \in \ell^{\mathbb{N}}$, then a daughter cell with initial size y is born and the size of this daughter cell evolves according to the process $\exp(A_{vr\ell^{\mathbb{N}}})$, where $A_{vr\ell^{\mathbb{N}}}$ is the process associated with $vr\ell^{\mathbb{N}} := \{vrw \ : \ w \in \ell^{\mathbb{N}}\}$ as in (3.2.7). Note that the process $\exp(A_{vr\ell^{\mathbb{N}}})$ has the same distribution as $y \exp(\xi)$. This daughter cell evolves independently of the other daughter cells, is killed at the first instant when her size drops suddenly by factor smaller than $\mathrm{e}^{\mathtt{d}}$, and gives birth to grand-daughter cells each time her size drops suddenly by factor smaller than $\mathrm{e}^{\bar{\mathtt{d}}}$ (note that $\mathrm{e}^{\bar{\mathtt{d}}} > \mathrm{e}^{\mathtt{d}}$) before being killed (with killing time included). We continue so on and so forth to obtain the higher generations. So we conclude that $\exp(\hat{\mathbf{L}})$ indeed has the same law as a truncated cell system associated with $X = \exp(\xi)$.

On the other hand, using the point of view that $\mathbf{L}$ is a BBLP with characteristics $(\sigma, c_b, \Lambda_{\mathbf{L}}, k, \Pi_{\mathbf{L}})$, since $D_{vr} > \mathtt{d}$ always holds by the construction, we may equivalently view $\hat{\mathbf{L}}$ as the system obtained from $\mathbf{L}$ by suppressing for each $v \in \mathcal{U}_2$ the smaller child D'_{vr} whenever $D'_{vr} \leq \mathtt{d}$. We hence deduce from Lemma 3 in [19] that $\hat{\mathbf{L}}$ is a BBLP with characteristics $(\sigma, c_b, \Lambda_{\mathbf{L}}^{\{\mathtt{d}\}}, k, \Pi_{\mathbf{L}}^{\{\mathtt{d}\}})$, where $\Lambda_{\mathbf{L}}^{\{\mathtt{d}\}}$ and $\Pi_{\mathbf{L}}^{\{\mathtt{d}\}}$ are derived from $\Lambda_{\mathbf{L}}$ and $\Pi_{\mathbf{L}}$ as in (3.2.5). We check that $(\Lambda_{\mathbf{L}}^{\{\mathtt{d}\}}, \Pi_{\mathbf{L}}^{\{\mathtt{d}\}}) = (\mathbf{1}_{\{[\bar{\mathtt{d}},0)\}} \Pi_b, \mathbf{1}_{\{[-\log 2, \bar{\mathtt{d}})\}} \Pi_b)$, so the two BBLPs $\hat{\mathbf{L}}$ and $\mathbf{Z}^{\mathtt{d}}$ have the same characteristics, which ends the proof. □

3.3 Markovian growth-fragmentation processes and bifurcators

In this section, we shall extend the notion of bifurcator to general cell processes and further establish a sufficient condition for different Markovian growth-fragmentations to have the same distribution, which finally orients us toward the proofs of Theorem 3.1.1 and Theorem 3.1.2. Let us first present a sufficient condition for non-explosion of growth-fragmentations, which slightly generalizes the approach in [20].

3.3.1 A sufficient condition for non-explosion

A Feller process $X = (X(t), t \geq 0)$ is called a **cell process**, if it has càdlàg path on $(0, \infty) \cup \{\partial\}$ with no positive jumps. We refer to ∂ as a cemetery point and denote the lifetime of X by $\zeta := \inf \{t \geq 0 \ : \ X(t) = \partial\} \in [0, \infty]$. For every $x \geq 0$ we write P_x for the law of X with initial value $X(0) = x$ and E_x for mathematical expectation under P_x.

As we have discussed in Section 3.2.3, to study the growth-fragmentation associated with X, we first want an ordering of the jumps of X, which is necessary to rigorously build a cell system driven by X. Furthermore, we need a sufficient condition for the *non-explosion* of the cell system, that is for every $t \geq 0$ the multiset of the sizes of all cells alive at time t is locally finite. For these purposes, we henceforth suppose the following hypothesis for X, which is reminiscent of that in Theorem 1 in [20].

[H] There exists a measurable function $f\colon [0,\infty) \times ([0,\infty) \cup \{\partial\}) \to [0,\infty)$, with $f(r,\partial) \equiv 0$ and $f(r,0) \equiv 0$ for every $r \geq 0$, which fulfills

$$\inf_{r<l, x>a} f(r,x) > 0, \qquad \text{for every } a, l > 0, \tag{3.3.1}$$

such that for every $x > 0$ and every $s, t \geq 0$, there is

$$E_x\left[f(s+t, X(t)) + \sum_{0\leq r\leq t} f(s+r, -\Delta X(r))\right] \leq f(s,x).$$

Example 3.3.1. *For $x > 0$, let P_x be the law of the homogeneous cell process $X^{(0)} = x\exp(\xi)$. Fix $q \geq 2$ and $K \geq \kappa(q)$, we have by an analogue of (3.2.3) that for every $x > 0$ and every $s, t \geq 0$*

$$\begin{aligned} &E_x\left[X^{(0)}(t)^q \mathrm{e}^{-K(t+s)} + \sum_{0\leq r\leq t} |\Delta X^{(0)}(r)|^q \mathrm{e}^{-K(r+s)}\right] \\ &= \left(\mathrm{e}^{(\Phi(q)-K)t} + \frac{\kappa(q)-\Phi(q)}{K-\Phi(q)}(1-\mathrm{e}^{(\Phi(q)-K)t})\right) x^q \mathrm{e}^{-Ks} \leq x^q \mathrm{e}^{-Ks}. \end{aligned}$$

So $X^{(0)}$ satisfies **[H]** *with the function $f(t,x) = x^q \mathrm{e}^{-Kt}$.*

From now on we fix a function f such that **[H]** holds for X. In particular **[H]** entails that for every $x > 0$

$$\sum_{r\geq 0} f(r, -\Delta X(r)) < \infty \quad P_x\text{-almost surely.}$$

Hence we may naturally enumerate the jump times of X by listing them in a sequence $(t_i)_{i\in\mathbb{N}}$ such that $(f(t_i, -\Delta X(t_i)))_{i\in\mathbb{N}}$ is decreasing, and thus reproduce the construction in Section 3.2.3 to build a cell system $\mathcal{X} := (\mathcal{X}_u,\ u \in \mathcal{U})$ driven by X, starting from an ancestor of initial size $x > 0$, with birth times b_u and life lengths $\zeta_u := \inf\{t \geq 0 \ :\ \mathcal{X}_u(t) = \partial\}$. Denote the sizes of the cells alive at time $t \geq 0$ by the multiset

$$\mathbf{X}(t) := \{\!\{\mathcal{X}_u(t-b_u) \ :\ u \in \mathcal{U}, b_u \leq t < b_u + \zeta_u\}\!\},$$

then $(\mathbf{X}(t), t \geq 0)$ is a **growth-fragmentation process driven by** X. We write $\mathcal{P}_x$ for the law of $\mathcal{X}$ and $\mathbf{P}_x$ for the law of $\mathbf{X}$ under $\mathcal{P}_x$. It is intuitively clear that the law of $\mathbf{X}$ is independent of the enumeration method.

For every non-negative measurable function $h : (0, \infty) \to [0, \infty)$ and every multiset $\mathcal{I}$ with elements in $(0, \infty)$, introduce the notation

$$\langle \mathcal{I}, h \rangle := \sum_{y \in \mathcal{I}} h(y) \in [0, \infty].$$

Let us define for every $s \geq 0$ a space $\mathcal{M}_f^s$: a multiset $\mathcal{I}$ is in $\mathcal{M}_f^s$, if $\mathcal{I}$ has elements in $(0, \infty)$ and $\langle \mathcal{I}, f(s, \cdot) \rangle < \infty$.

Lemma 3.3.2. *Suppose that X satisfies* [H] *with a function f. Then we have for every $x > 0$ that*

$$\mathbf{E}_x \left[\langle \mathbf{X}(t), f(s+t, \cdot) \rangle \right] \leq f(s, x), \quad \textit{for all } t, s \geq 0,$$

where $\mathbf{E}_x$ denotes the mathematical expectation under $\mathbf{P}_x$. So we have $\mathbf{P}_x$-almost surely $\mathbf{X}(t) \in \mathcal{M}_f^t$.

Lemma 3.3.2 encompasses Theorem 1 in [20] for the case when f only depends on the x variable, i.e. $f(t, x) \equiv f(x)$ for every $x, t \geq 0$. In that case f is a so-called *excessive function* for $\mathbf{X}$. In the same spirit, we may refer to f as a *time-dependent excessive function* for $\mathbf{X}$.

Proof. The proof is an adaptation of arguments of Theorem 1 in [20]. We may assume that $\mathbf{X}$ is associated with a cell system $\mathcal{X}$ of law $\mathcal{P}_x$ and write $\mathcal{E}_x$ for mathematical expectation under $\mathcal{P}_x$. We will prove that the sequence

$$\Sigma(i) := \sum_{|u| \leq i, b_u \leq t} f(s+t, \mathcal{X}_u(t - b_u)) + \sum_{|v| = i, b_v \leq t} \sum_{b_v \leq r \leq t} f(s+r, -\Delta \mathcal{X}_v(r - b_v)), \quad i \in \mathbb{N}$$

is a non-negative super-martingale, then $\Sigma(\infty) = \lim_{i \to \infty} \Sigma(i)$ exists almost surely and $\Sigma(\infty) \geq \langle \mathbf{X}(t), f(s+t, \cdot) \rangle$. We thus deduce from Fatou's lemma that

$$\mathbf{E}_x \left[\langle \mathbf{X}(t), f(s+t, \cdot) \rangle \right] \leq \mathcal{E}_x [\Sigma(0)] = E_x \left[f(s+t, X(t)) + \sum_{0 \leq r \leq t} f(s+r, -\Delta X(r)) \right] \leq f(s, x),$$

where the last inequality derives from [H].

So it remains to prove that $\Sigma(i)$ is a super-martingale. For every v with $|v| = i$, given $\mathcal{F}_{i-1} := \sigma(\mathcal{X}_u, |u| \leq i-1)$ we have by [H] that

$$\mathcal{E}_x \left[f(s+t, \mathcal{X}_v(t - b_v)) + \sum_{b_v \leq r \leq t} f(s+r, -\Delta \mathcal{X}_v(r - b_v)) \,\middle|\, \mathcal{F}_{i-1} \right] \leq f(s + b_v, \mathcal{X}_v(0)).$$

Summing over v of i-th generation on the event $\{t \geq b_v\}$, we get that

$$\mathcal{E}_x \left[\sum_{|v|=i, b_v \leq t} f\left(s+t, \mathcal{X}_v(t-b_v)\right) + \sum_{|v|=i, b_v \leq t} \sum_{b_v \leq r \leq t} f\left(s+t, -\Delta \mathcal{X}_v(r-b_v)\right) \;\middle|\; \mathcal{F}_{i-1} \right]$$
$$\leq \sum_{|v|=i, b_v \leq t} f\left(s+b_v, \mathcal{X}_v(0)\right) = \sum_{|u|=i-1, b_u \leq t} \sum_{b_u \leq r \leq t} f\left(s+r, -\Delta \mathcal{X}_u(r-b_u)\right).$$

Adding $\sum_{|u| \leq i-1, b_u \leq t} f\left(s+t, \mathcal{X}_u(t-b_u)\right)$ to both sides of inequality, we conclude that

$$\mathcal{E}_x \left[\Sigma(i) \mid \mathcal{F}_{i-1}\right] \leq \Sigma(i-1),$$

which means that $\Sigma(i)$ is a super-martingale. $\square$

Let $\mathcal{M}_+$ be the class of all multisets $\mathcal{I}$ on $(0, \infty)$, which has only finitely many elements in $[a, \infty)$ for every $a > 0$. Note that each $\mathcal{I} \in \mathcal{M}_+$ corresponds to a Radon measure (in the sense of Remark 3.2.9) and (3.3.1) ensures that $\mathcal{M}_f^s \subset \mathcal{M}_+$ for every $s \geq 0$. On account of Lemma 3.3.2, we can hence view the growth-fragmentation $\mathbf{X}$ as a stochastic process with values in $\mathcal{M}_+$, which means that $\mathbf{X}$ does not explode. The space $\mathcal{M}_+$ is endowed with the following topology:

Definition 3.3.3. *We denote the cardinality of a multiset $\mathcal{J}$ by $|\mathcal{J}|$. A sequence $(\mathcal{I}_n)_{n \in \mathbb{N}} \in \mathcal{M}_+$ converges to $\mathcal{I} \in \mathcal{M}_+$ if and only if for all $r \in (0, \infty)$ such that $\mathcal{I} \cap \{r\} = \emptyset$ there is $|\mathcal{I}_n \cap [r, \infty)| \to |\mathcal{I} \cap [r, \infty)|$.*

The advantage of endowing $\mathcal{M}_+$ with this topology is that it is a Polish space (homeomorphic to a complete and separable metric space), see Theorem 2.1 and Theorem 2.2 in [63]. It is known from Lemma 2.1 in [63] that convergence in $\mathcal{M}_+$ implies vague convergence. See [63] for more properties of $\mathcal{M}_+$.

We next introduce a *truncate* operation on $\mathcal{X}$ tailored for our future purpose, which is different from the one in the proof of Proposition 3.2.15. For every $\epsilon > 0$, we obtain a truncated system $\mathcal{X}^{[\epsilon]} = (\mathcal{X}_u^{[\epsilon]}, u \in \mathcal{U})$, by killing each cell process at the first time $s \geq 0$ when its size is less than or equal to ϵ, together with its future (born at time $> s$) descendants. Specifically, let us denote for every $u \in \mathcal{U}$ its ancestral lineage by $A_u := (A_u(t), t \geq 0)$, i.e.

$$A_u(t) := \sum_{n \leq |u|-1} \mathcal{X}_{[u]_n}(t - b_{[u]_n}) \mathbf{1}_{\left\{b_{[u]_n} \leq t < b_{[u]_{n+1}}\right\}} + \mathcal{X}_u(t-b_u) \mathbf{1}_{\{b_u \leq t\}}, \quad t \geq 0,$$

where $[u]_n$ denotes u's ancestor at the n-th generation for all $n \leq |u|$, then we have that

$$\mathcal{X}_u^{[\epsilon]}(t) := \begin{cases} \mathcal{X}_u(t), & \text{if } \inf_{0 \leq r \leq t+b_u} A_u(r) > \epsilon. \\ \partial, & \text{otherwise.} \end{cases} \tag{3.3.2}$$

Let $\mathbf{X}^{[\epsilon]}$ be the point process on $(0,\infty)$ associated with $\mathcal{X}^{[\epsilon]}$:

$$\mathbf{X}^{[\epsilon]}(t) = \{\!\{\mathcal{X}_u^{[\epsilon]}(t-b_u):\ u\in\mathcal{U}, b_u\le t, \mathcal{X}_u^{[\epsilon]}(t-b_u)\neq\partial\}\!\}, \quad t\ge 0.$$

Lemma 3.3.4. *Suppose that X satisfies* [H]. *Then for every $x\ge 0$ and every $t\ge 0$, under the topology of $\mathcal{M}_+$ the multiset $\mathbf{X}^{[\epsilon]}(t)$ converges $\mathbf{P}_x$-almost surely to $\mathbf{X}(t)$ as $\epsilon\downarrow 0+$.*

Proof. We first note that if a cell $u\in\mathcal{U}$ is alive at time $t\ge 0$ with $\mathcal{X}_u(t-b_u)>0$, then $\mathbf{P}_x$-almost surely its ancestral lineage has a size bounded away from 0 before time t, i.e. $\inf_{0\le r\le t} A_u(r)>0$. So there exists $\epsilon>0$ small enough such that $\mathcal{X}_u^{[\epsilon]}(r-b_u)=\mathcal{X}_u(r-b_u)$ for all $b_u\le r\le t$, and we have $\mathbf{P}_x$-almost surely

$$\lim_{\epsilon\to 0+}\mathcal{X}_u^{[\epsilon]}(t-b_u)\mathbf{1}_{\{t\ge b_u\}} = \mathcal{X}_u(t-b_u)\mathbf{1}_{\{t\ge b_u\}}.$$

We hence obtain by the monotone convergence that for every $a>0$, $\mathbf{P}_x$-almost surely

$$\lim_{\epsilon\to 0}\sum_{u\in\mathcal{U}}\mathbf{1}_{\left\{\mathcal{X}_u^{[\epsilon]}(t-b_u)\ge a\right\}}\mathcal{X}_u^{[\epsilon]}(t-b_u)\mathbf{1}_{\{t\ge b_u\}} = \sum_{u\in\mathcal{U}}\lim_{\epsilon\to 0}\mathbf{1}_{\left\{\mathcal{X}_u^{[\epsilon]}(t-b_u)\ge a\right\}}\mathcal{X}_u^{[\epsilon]}(t-b_u)\mathbf{1}_{\{t\ge b_u\}},$$

which means that $\mathbf{P}_x$-almost surely

$$\lim_{\epsilon\to 0}|\mathbf{X}^{[\epsilon]}(t)\cap[a,\infty)| = |\mathbf{X}(t)\cap[a,\infty)|.$$

So we conclude that $\mathbf{X}^{[\epsilon]}(t)$ converges $\mathbf{P}_x$-almost surely to $\mathbf{X}(t)$ in $\mathcal{M}_+$. $\square$

We observe that the truncated system $\mathcal{X}^{[\epsilon]}$ has a discrete temporal branching structure, since for each càdlàg process the set of jump times with sizes of jumps $<-\epsilon$ is discrete. By the same arguments as the proof of Proposition 2 in [20], we deduce from this observation and Lemma 3.3.4 that $\mathbf{X}$ has the temporal branching property. To describe this property, let us define a family $(\rho_{s,t}, t\ge s\ge 0)$, where each $\rho_{s,t}$ is a probability kernel from $\mathcal{M}_f^s$ to $\mathcal{M}_f^t$, in the following way. Given a multiset $\mathcal{J}\in\mathcal{M}_f^s$, we may construct a family of independent random multisets $(\mathcal{I}_y, y\in\mathcal{J})$, such that each $\mathcal{I}_y$ has the law of $\mathbf{X}(t-s)$ under $\mathbf{P}_y$. Define by the sum of the multisets $\mathcal{J}^t := \biguplus_{y\in\mathcal{J}}\mathcal{I}_y$, then $\mathcal{J}^t\in\mathcal{M}_f^t$, since it follows from Lemma 3.3.2 that

$$\left\langle \mathcal{J}^t, f(t,\cdot)\right\rangle = \sum_{y\in\mathcal{J}}\mathbf{E}_y\left[\left\langle \mathbf{X}(t-s), f(t,\cdot)\right\rangle\right] \le \sum_{y\in\mathcal{J}} f(s,y) = \left\langle \mathcal{J}, f(s,\cdot)\right\rangle < \infty.$$

We hence define $\rho_{s,t}(\mathcal{J},\cdot)$ by the law of $\mathcal{J}^t$.

Proposition 3.3.5 (Temporal branching property). *Suppose that X satisfies* [H] *with a function f. Then for every $t\ge s\ge 0$ and every $x>0$, the conditional distribution of $\mathbf{X}(t)$ under $\mathbf{P}_x$ given $(\mathbf{X}(r), 0\le r\le s)$ is $\rho_{s,t}(\mathbf{X}(s),\cdot)$.*

Remark 3.3.6. *One may easily extend the analysis in this section to time-inhomogeneous Markov processes. Let X is a time-inhomogeneous cell process and write $P_{s,x}$ for the law of X starting at time $s \geq 0$ with initial size $x \geq 0$. Then the counterpart of condition* **[H]** *is that there exists a function f that satisfies (3.3.1) and for every $x > 0$ and every $s \geq 0$,*

$$E_{s,x}\left[f(t, X(t)) + \sum_{s \leq r \leq t} f(r, -\Delta X(r))\right] \leq f(s, x), \quad \textit{for all } t \geq s,$$

where $\mathrm{E}_{s,x}$ means mathematical expectation under $P_{s,x}$. Under this condition, one may easily build a cell system driven by X (with the life path of each $\mathcal{X}_u$ scaled by the universal time) and check that the system does not explode by an analogue of Lemma 3.3.2. Details shall be left to interested readers.

3.3.2 Bifurcators

For every $x > 0$, let P_x and Q_x be respectively the laws of two cell processes X and Y, both starting from x. We now give a formal definition of bifurcators of cell processes, which extends both Definition 2 by Pitman and Winkel [70] and the present Definition 3.2.6 for homogeneous cell processes.

Definition 3.3.7. *A bivariate process (X', Y') is called a* **bifurcator** *of branches X and Y, if it satisfies the following properties:*

(i) For every $x > 0$, write $\mathbb{P}_x$ for the joint distribution of (X', Y') with $X'(0) = Y'(0) = x$. Under $\mathbb{P}_x$, each component X' and Y' has the law P_x and Q_x respectively, that is, the two marginal distributions of $\mathbb{P}_x$ are P_x and Q_x.

(ii) Let $\tau := \inf\{t \geq 0 : X'(t) \neq Y'(t)\}$. For every $x > 0$, conditionally on $\{\tau < \infty\}$, there is

$$X'(\tau) + Y'(\tau) = X'(\tau-) = Y'(\tau-), \quad \mathbb{P}_x - a.s. \tag{3.3.3}$$

(iii) (Asymmetric Markov branching property) For every $x > 0$, the process

$$(X'(t), Y'(t), \mathbf{1}_{\{\tau > t\}})_{t \geq 0}$$

under $\mathbb{P}_x$ is Markovian. Specifically, conditionally given $\tau > t$, the process $(X'(r+t), Y'(r+t))_{r \geq 0}$ has distribution $\mathbb{P}_{X'(t)}$; conditionally given $\tau \leq t$, $(X'(r+t), Y'(r+t))_{r \geq 0}$ is a pair of independent processes of respective laws $P_{X'(t)}$ and $Q_{Y'(t)}$.

If such a bifurcator (X', Y') exists, then we say X **and** *Y* **can be coupled to form a bifurcator**.

Remark 3.3.8. *We know from (3.3.3) that if $\tau < \infty$, then (3.3.3) implies that τ is a jump time of both X' and Y', which is almost surely strictly positive and strictly smaller than the lifetimes of X' and Y'. Define a filtration $(\mathcal{G}_t)_{t\geq 0}$ by the usual augmentation of $\sigma(X'(r), Y'(r),\ 0 \leq r \leq t)$, note that τ is a $(\mathcal{G}_t)$-stopping time and each component X' or Y' satisfies the strong Markov property.*

We next state a sufficient condition for growth-fragmentations based on different cell processes to have the same distribution, which is the main purpose of this work. Suppose that [H] holds for both X and Y, then we know from the preceding subsection that we can construct two non-explosive growth-fragmentations $\mathbf{X}$ and $\mathbf{Y}$ associated with X and Y respectively. Note that [H] entails that for every $x > 0$ and every $s \geq 0$,

$$E_x\left[\sum_{r\geq 0} f(s+r, -\Delta X(r))\right] \leq f(s,x).$$

However, we shall need a stronger inequality and make the following assumption:

[Hη] For a certain $\eta \in (0,1)$, there exists a function g that satisfies

$$\inf_{r<l, x>a} g(r,x) > 0, \qquad \text{for every } a, l > 0, \tag{3.3.4}$$

such that for every $x > 0$ and every $s \geq 0$,

$$E_x\left[\sum_{r\geq 0} g(s+r, -\Delta X(r))\right] \leq \eta g(s,x).$$

Theorem 3.3.9. *Let X and Y be two cell processes that both satisfy [H] and [Hη]. Suppose that X and Y can be coupled to form a bifurcator, then for every $x > 0$, two Markovian growth-fragmentations $\mathbf{X}$ and $\mathbf{Y}$ driven respectively by X and Y, both starting from x, have the same finite-dimensional distributions.*

Before proceeding to the proof, let us give a consequence of [Hη].

Lemma 3.3.10. *If X satisfies [Hη] with g and η, then for every $x > 0$*

$$\lim_{n\to\infty} \sum_{|u|=n} g(b_u, \mathcal{X}_u(0)) = 0 \qquad \mathcal{P}_x\text{-almost surely.}$$

Proof of Lemma 3.3.10. Similar to the proof of Lemma 3.3.2, we can easily check that

$$M_n := \eta^{-n} \sum_{|u|=n+1} g(b_u, \mathcal{X}_u(0)) = \eta^{-n} \sum_{|u|=n} \sum_{r\geq 0} g(b_u + r, -\Delta\mathcal{X}_u(r))$$

is a non-negative super-martingale, so M_n converges $\mathcal{P}_x$-almost surely to a limit M_∞. As a consequence, $\eta^n M_n$ converges $\mathcal{P}_x$-almost surely to zero, which is our claim. $\square$

Under assumption **[Hη]**, suppose that there exists $\bar{u} \in \partial\mathcal{U}$ such that $\lim_{n\to\infty} b_{[\bar{u}]_n} < \infty$, where $[\bar{u}]_n$ denotes the ancestor of $\bar{u}$ at the n-th generation. Then since g satisfies (3.3.4), it follows from Lemma 3.3.10 that $\limsup_{n\to\infty} \mathcal{X}_{[\bar{u}]_n}(0) = 0$.

3.3.3 Proof of Theorem 3.3.9

Let us briefly explain the idea of the proof. Fix $x > 0$, let $\mathcal{X}$ and $\mathcal{Y}$ be cell systems associated with X and Y respectively, with respective laws $\mathcal{P}_x$ and $\mathcal{Q}_x$. For every $\epsilon > 0$, let $\mathbf{X}^{[\epsilon]}$ be the process associated with the truncated cell system $\mathcal{X}^{[\epsilon]}$ derived from $\mathcal{X}$ as in (3.3.2), by killing each cell together with its future descendance when its size becomes less than or equal to ϵ. Similarly we define $\mathbf{Y}^{[\epsilon]}$. We shall prove for every $\epsilon > 0$ that $\mathbf{X}^{[\epsilon]}$ under $\mathcal{P}_x$ has the same law as $\mathbf{Y}^{[\epsilon]}$ under $\mathcal{Q}_x$. Then letting $\epsilon \to 0+$, we conclude from Lemma 3.3.4 that $\mathbf{X}$ and $\mathbf{Y}$ have the same finite-dimensional distributions.

Let us fix an arbitrary $\epsilon > 0$. To prepare for the proof that $\mathbf{X}^{[\epsilon]}$ and $\mathbf{Y}^{[\epsilon]}$ have the same law, we construct a family of bivariate processes $((X_v, Y_v), v \in \mathcal{U}_2)$ (recall that $\mathcal{U}_2 = \bigcup_{n\in\mathbb{N}} \{\ell, r\}^n$ is the binary tree) in the following way. Since X and Y can be coupled to form a bifurcator, there exists a bifurcator with distribution $(\mathbb{P}_y, y > 0)$, whose marginal distributions are P_y and Q_y under $\mathbb{P}_y$. Then we let $(X_\emptyset, Y_\emptyset)$ be a bifurcator with law $\mathbb{P}_x$ and write $\beta_\emptyset := 0$ for the birth time of $\emptyset$. Suppose by induction that we have built for a certain $v \in \mathcal{U}_2$ a bifurcator (X_v, Y_v) with birth time β_v. Write $\tau_v := \inf\{t \geq 0 : X_v(t) \neq Y_v(t)\}$ for the switching time of this bifurcator, $T_v^X := \inf\{t \geq 0 : \ X_v(t) \leq \epsilon\}$ for the first time when X_v is smaller than ϵ, and $\tilde{T}_v^X := \inf\{t \geq 0 : \ -\Delta X_v(t) > \epsilon\}$ for the first time when X_v has a jump of size greater than ϵ, then we define the lifetime of v by

$$\lambda_v := \tau_v \wedge T_v^X \wedge \tilde{T}_v^X,$$

then λ_v is a $(\mathcal{G}_t^v)$-stopping time, where $\mathcal{G}_t^v$ is the usual augmentation of $\sigma((X_v(r), Y_v(r)),\ 0 \leq r \leq t)$. At the lifetime λ_v, we distinguish the following two situations.

- If $\lambda_v = T_v^X < \tilde{T}_v^X \wedge \tau_v$ or $\lambda_v = \infty$, then v is **killed** at its lifetime λ_v. Further, we agree that for every $w \in \mathcal{U}_2 \setminus \{\emptyset\}$, vw is also killed, with $\beta_{vw} = \infty$, $\lambda_{vw} = 0$ and $X_{vw} \equiv Y_{vw} \equiv \partial$. As $\tau_v \wedge \tilde{T}_v^X$ is almost surely strictly positive, this situation also covers the case when $\lambda_v = 0$ (if and only if $T_v^X = 0$, i.e. $X_v(0) \leq \epsilon$).

- Otherwise, v **branches** at its lifetime λ_v, giving birth to two independent bifurcators $(X_{v\ell}, Y_{v\ell})$ and (X_{vr}, Y_{vr}) with respective distributions $\mathbb{P}_{a_{v\ell}}$ and $\mathbb{P}_{a_{vr}}$, where

$$(a_{v\ell}, a_{vr}) := (X_v(\lambda_v), -\Delta X_v(\lambda_v)). \tag{3.3.5}$$

Set their birth time by $\beta_{v\ell} = \beta_{vr} := \beta_v + \lambda_v$. We further **mark** v if $\lambda_v = \tau_v \le T_v^X \wedge \tilde{T}_v^X$ (we also say that we mark the branching event at the death of v), so v is **non-marked** if $\lambda_v = \tilde{T}_v^X < \tau_v$. Using the junction relation (3.3.3) of the bifurcator, we also have

$$(a_{v\ell}, a_{vr}) = \begin{cases} (-\Delta Y_v(\lambda_v), Y_v(\lambda_v)), & \text{if } v \text{ is marked,} \\ (Y_v(\lambda_v), -\Delta Y_v(\lambda_v)), & \text{if } v \text{ is non-marked.} \end{cases} \tag{3.3.6}$$

Note that if v is non-marked, then $a_{vr} > \epsilon$ always holds; but if v is marked, then it is possible that $a_{vr} \le \epsilon$, which means that vr is immediately killed and $\lambda_{vr} = 0$. In both marked and non-marked cases, it is possible that $a_{v\ell} \le \epsilon$ and $v\ell$ is immediately killed with $\lambda_{v\ell} = 0$.

We continue so on and so forth to construct all generations of the family $((X_v, Y_v), v \in \mathcal{U}_2)$ and finally define a process

$$\mathbf{W}_{(X,Y)}(t) := \{\!\!\{ X_v(t - \beta_v) \ : \ v \in \mathcal{U}_2, \beta_v \le t < \beta_v + \lambda_v \}\!\!\}, \qquad t \ge 0.$$

Note by construction that every element of $\mathbf{W}_{(X,Y)}(t)$ is larger than ϵ. A notable feature of this system is that, roughly speaking, $\mathbf{W}_{(X,Y)}$ is symmetric, i.e. its law is invariant under the permutation of labels X and Y.

Lemma 3.3.11. *$\mathbf{W}_{(X,Y)}$ has the same law as $\mathbf{W}_{(Y,X)}$.*

Proof. Given the family $((X_v, Y_v), v \in \mathcal{U}_2)$ constructed as above, let us define recursively a bijection $h : \mathcal{U}_2 \to \mathcal{U}_2$ with $h(\emptyset) := \emptyset$, such that for every $v \in \mathcal{U}_2$ we have $(h(v\ell), h(vr)) := (h(v)r, h(v)\ell)$ if $h(v)$ is marked, and $(h(v\ell), h(vr)) := (h(v)\ell, h(v)r)$ if v is non-marked or v is killed. We next describe the dynamics of $((Y'_v, X'_v), v \in \mathcal{U}_2) := ((Y_{h(v)}, X_{h(v)}), v \in \mathcal{U}_2)$ as a bivariate system generated by the bifurcator (Y, X). Specifically, define T_v^Y and $\tilde{T}_v^Y$ for Y_v in the same way as T_v^X and $\tilde{T}_v^X$, then the lifetime of each (Y'_v, X'_v) is $\lambda'_v := \tau_{h(v)} \wedge T^Y_{h(v)} \wedge \tilde{T}^Y_{h(v)}$, which is equal to $\lambda_{h(v)}$. Indeed, since $X_{h(v)}(t) = Y_{h(v)}(t)$ for all $t < \tau_{h(v)}$, we find that

- If $\lambda_{h(v)} = T^X_{h(v)} < \tilde{T}^X_{h(v)} \wedge \tau_{h(v)}$ or $\lambda_{h(v)} = \infty$, then $T^Y_{h(v)} = T^X_{h(v)}$ and $T^Y_{h(v)} < \tilde{T}^Y_{h(v)} \wedge \tau_{h(v)}$;
- If $\lambda_{h(v)} = \tau_{h(v)} \le T^X_{h(v)} \wedge \tilde{T}^X_{h(v)}$, then $\tau_{h(v)} \le T^Y_{h(v)} \wedge \tilde{T}^Y_{h(v)}$;
- if $\lambda_{h(v)} = \tilde{T}^X_{h(v)} < \tau_{h(v)}$, then $\tilde{T}^Y_{h(v)} = \tilde{T}^X_{h(v)} < \tau_{h(v)}$ and $\tilde{T}^Y_{h(v)} \le T^Y_{h(v)}$.

At the lifetime $\lambda_{h(v)}$, v is killed in the first case; in the other two cases, v generates two independent bifurcators $(Y'_{v\ell}, X'_{v\ell})$ and (Y'_{vr}, X'_{vr}) of respective laws $\mathbb{P}_{a_{h(v\ell)}}$ and $\mathbb{P}_{a_{h(vr)}}$. It follows from (3.3.6) and the construction of h that for every $v \in \mathcal{U}_2$:

$$(a_{h(v\ell)}, a_{h(vr)}) = (Y_{h(v)}(\lambda_{h(v)}), -\Delta Y_{h(v)}(\lambda_{h(v)})).$$

Define for every $t \geq 0$ that

$$\begin{aligned}\mathbf{W}'(t) :=& \{\!\!\{ Y'_v(t-\beta'_v):\ v \in \mathcal{U}_2, \beta'_v \leq t < \beta'_v + \lambda'_v \}\!\!\} \\ =& \{\!\!\{ Y_{h(v)}(t-\beta_{h(v)}):\ v \in \mathcal{U}_2, \beta_{h(v)} \leq t < \beta_{h(v)} + \lambda_{h(v)} \}\!\!\},\end{aligned}$$

then the process $(\mathbf{W}'(t), t \geq 0)$ is a copy of $\mathbf{W}_{(Y,X)}$. On the other hand, since h is a bijection and recall that for every $v \in \mathcal{U}_2$, $X_v(t) = Y_v(t)$ for all $t < \lambda_v$, then clearly $\mathbf{W}' = \mathbf{W}_{(X,Y)}$. □

We next consider the process associated with the left-most branch $\ell^{\mathbb{N}} = \{\emptyset, \ell, \ell\ell, \ldots\}$, that is

$$A_{\ell^{\mathbb{N}}}(t) := \sum_{n\geq 0} \mathbf{1}_{\{\beta_{\ell^n} \leq t < \beta_{\ell^n}+\lambda_{\ell^n}\}} X_{\ell^n}(t-\beta_{\ell^n}) = \sum_{n\geq 0} \mathbf{1}_{\{\beta_{\ell^n} \leq t < \beta_{\ell^n}+\lambda_{\ell^n}\}} Y_{\ell^n}(t-\beta_{\ell^n}), \quad t \geq 0. \tag{3.3.7}$$

where $\ell^n := (\ell\ell\ldots\ell) \in \{\ell, r\}^n$ and $\ell^0 := \emptyset$. Let $N := \inf\{n \in \mathbb{N}:\ \ell^n \text{ is killed}\}$, with convention $\inf\{\emptyset\} = \infty$. If $N < \infty$, then $A_{\ell^{\mathbb{N}}}(t) = \partial$ for all $t \geq \beta_{\ell^N} + \lambda_{\ell^N}$. By concatenating $A_{\ell^{\mathbb{N}}}$ with the segment of X_{ℓ^N} after its lifetime λ_{ℓ^N}, we define

$$A^X_{\ell^{\mathbb{N}}}(t) := \begin{cases} A_{\ell^{\mathbb{N}}}(t), & t < \beta_{\ell^N} + \lambda_{\ell^N}. \\ X_{\ell^N}(t-\beta_{\ell^N}),, & t \geq \beta_{\ell^N} + \lambda_{\ell^N}. \end{cases} \tag{3.3.8}$$

We agree that $A^X_{\ell^{\mathbb{N}}} = A_{\ell^{\mathbb{N}}}$ if $N = \infty$.

Lemma 3.3.12. *Suppose that [**H**η] holds for both X and Y. Then the process $A^X_{\ell^{\mathbb{N}}}$ has the law of P_x (the law of X starting from x), and the process derived from $A^X_{\ell^{\mathbb{N}}}$ by killing at $\zeta^X_{\ell^{\mathbb{N}}} := \inf\left\{t \geq 0:\ A^X_{\ell^{\mathbb{N}}}(t) \leq \epsilon\right\}$ is $A_{\ell^{\mathbb{N}}}$.*

Proof. It should be intuitive that $A^X_{\ell^{\mathbb{N}}}$ has the law of P_x because of the construction (3.3.5) and the strong Markov property of X; however, it is a priori possible that none of $\ell^{\mathbb{N}}$ is killed and their birth times accumulate to a finite limit, i.e. $N = \infty$ and $\lim_{n\to\infty} \beta_{\ell^n} < \infty$, then $A^X_{\ell^{\mathbb{N}}}$ is killed at this limit time, thus does not have the law of P_x. We shall prove that this case does not happen, thanks to the assumption [**H**η]. Therefore, almost surely there are only two possible situations: either $N < \infty$, or $N = \infty$ & $\lim_{n\to\infty} \beta_{\ell^n} = \infty$, so we deduce from the strong Markov property of X that $A^X_{\ell^{\mathbb{N}}}$ indeed has the law of P_x. Further, we easily check that $\zeta^X_{\ell^{\mathbb{N}}} = \beta_{\ell^N} + \lambda_{\ell^N}$ when $N < \infty$ and $\zeta^X_{\ell^{\mathbb{N}}} = \infty$ when $N = \infty$, then the second part of the claim follows.

So it remains to prove that if $N = \infty$, which means that none of $(\ell^n)_{n\in\mathbb{N}}$ is killed, then $\lim_{n\to\infty} \beta_{\ell^n} = \infty$. We consider separately the following two situations.

In the first situation there are infinitely many marked elements in $\ell^{\mathbb{N}}$, and we list all of them in a sequence $(\ell^{n_i})_{i\in\mathbb{N}} \subset \ell^{\mathbb{N}}$ with $n_i \uparrow \infty$. Let $\mathcal{G}_n := \sigma(X_{\ell^j}, Y_{\ell^j}, j \leq n)$ and g_Y be a function such that [**H**η] holds for Y with $\eta_Y < 1$, then

$$M_i := \eta_Y^{-i} \sum_{r\geq 0} g_Y(\beta_{\ell^{n_i}} + r, -\Delta Y_{\ell^{n_i}}(r)), \quad i \in \mathbb{N}$$

is a non-negative $\mathcal{G}_{n_i}$-super-martingale. Indeed, consider the ancestral lineage of $\ell^{n_{i+1}}$ for the Y-side, shifted to the left by $\beta_{\ell^{n_i}\ell}$ ($\ell^{n_i}\ell$ means ℓ^{n_i+1}), that is

$$A^Y_{i+1}(t) := \sum_{n_i+1\leq k<n_{i+1}} \mathbf{1}_{\{\beta_{\ell^k}\leq t+\beta_{\ell^{n_i}\ell}<\beta_{\ell^k}+\lambda_{\ell^k}\}} Y_{\ell^k}(t+\beta_{\ell^{n_i}\ell}-\beta_{\ell^k})$$
$$+\mathbf{1}_{\{t+\beta_{\ell^{n_i}\ell}\geq\beta_{\ell^{n_{i+1}}}\}} Y_{\ell^{n_{i+1}}}(t+\beta_{\ell^{n_i}\ell}-\beta_{\ell^{n_{i+1}}}), \quad t\geq 0,$$

with $A^Y_{i+1}(0)=Y_{\ell^{n_i}\ell}(0)$. Then

$$M_{i+1}\leq \eta_Y^{-(i+1)}\sum_{r\geq 0} g_Y(\beta_{\ell^{n_i}\ell}+r,-\Delta A^Y_{i+1}(r)).$$

Observing that these segments are connected by only non-marked branching events and using (3.3.6), we hence deduce by the strong Markov property of Y that conditionally on $\mathcal{G}_{n_i}$, A^Y_i has distribution Q_y with $y:=Y_{\ell^{n_i}\ell}(0)$. As Y satisfies **[Hη]**, we have

$$\mathscr{E}\left[M_{i+1}\middle|\mathcal{G}_{n_i}\right]\leq \mathscr{E}\left[\eta_Y^{-(i+1)}\sum_{r\geq 0} g_Y(\beta_{\ell^{n_i}\ell}+r,-\Delta A^Y_{i+1}(r))\middle|\mathcal{G}_{n_i}\right]$$
$$\leq \eta_Y^{-i} g_Y(\beta_{\ell^{n_i}\ell},Y_{\ell^{n_i}\ell}(0)) = \eta_Y^{-i} g_Y(\beta_{\ell^{n_i}}+\lambda_{\ell^{n_i}},-\Delta Y_{\ell^{n_i}}(\lambda_{\ell^{n_i}}))\leq M_i,$$

where $\mathscr{E}$ denotes the mathematical expectation under the law $\mathscr{P}$ of the system $(X_v,Y_v,v\in\mathcal{U}_2)$, and the equality follows from (3.3.6) as ℓ^{n_i} is marked. We conclude that M_i is a non-negative super-martingale and hence M_i converges almost surely to a limit as $i\to\infty$. Multiplying the last display by $\eta_Y^{n_i}$, we have

$$g_Y(\beta_{\ell^{n_i}\ell},X_{\ell^{n_i}\ell}(0)) = g_Y(\beta_{\ell^{n_i}}+\lambda_{\ell^{n_i}},-\Delta Y_{\ell^{n_i}}(\lambda_{\ell^{n_i}}))\leq \eta_Y^i M_i\to 0 \quad \text{almost surely}.$$

As g_Y satisfies (3.3.4), it follows that in the event $\lim_{n\to\infty}\beta_{\ell^n}<\infty$, there is $\lim_{i\to\infty}X_{\ell^{n_i}\ell}(0)=0$. This is absurd as we have assumed that no element in $\ell^{\mathbb{N}}$ is killed.

In the second situation, there are infinitely many non-marked branching elements in $\ell^{\mathbb{N}}$. Consider for each $k\in\mathbb{N}$ the ancestral lineage of ℓ^k for the side of X, i.e.

$$A^X_{\ell^k}(t) := \sum_{n=0}^{k-1}\mathbf{1}_{\{\beta_{\ell^n}\leq t<\beta_{\ell^n\ell}\}}X_{\ell^n}(t-\beta_{\ell^n})+\mathbf{1}_{\{\beta_{\ell^k}\leq t\}}X_{\ell^k}(t-\beta_{\ell^k}), \quad t\geq 0.$$

Then for each $k\in\mathbb{N}$, we deduce from the strong Markov property of X that $A^X_{\ell^k}$ has law P_x. Let g_X be a function such that **[Hη]** holds for X with constant $\eta_X<1$, then

$$E_x\left[\sum_{r\geq 0} g_X(r,-\Delta A^X_{\ell^k}(r))\right]\leq g_X(0,x). \tag{3.3.9}$$

Suppose, by contradiction, that there exists a certain $M > 0$ such that with probability $p_M > 0$ there is $\lim_{n\to\infty} \beta_{\ell^n} < M$ and write $\inf_{t<M,y\geq\epsilon} g_X(t,y) =: c_{M,\epsilon} > 0$ as (3.3.4) holds for g_X. For every $k \in \mathbb{N}$, write m_k for the number of non-marked particles in the set $\{\ell^i, i < k\}$, then we get that

$$E_x \left[\sum_{r\geq 0} g_X(r, -\Delta A^X_{\ell^k}(r)) \right] \geq \mathscr{E} \left[\sum_{1\leq i\leq k-1} \mathbf{1}_{\{\ell^i \text{ is non-marked}\}} g_X(\beta_{\ell^i}, -\Delta X_{\ell^i}(\lambda_{\ell^i})) \right] \geq p_M m_k c_{M,\epsilon},$$

where the last inequality is obtained by restricting to the event $\lim_{n\to\infty} \beta_{\ell^n} < M$ and observing that $-\Delta X_{\ell^i}(\lambda_{\ell^i}) \geq \epsilon$ whenever ℓ^i is non-marked. Letting $k \to \infty$, we find a contradiction against (3.3.9) since $m_k \to \infty$. We have therefore proved the claim. □

Remark 3.3.13. *Given the system $((X_v, Y_v), v \in \mathcal{U}_2)$, let us define a branch $\bar{v} := (v_n \in \mathcal{U}_2)_{n\geq 0}$ with $v_0 := \emptyset$ by setting recursively $v_{n+1} = v_n\ell$ if v_n is non-marked and $v_{n+1} = v_n r$ if v_n is marked. Then the branch $A^Y_{\bar{v}}$ associated with the system $((X_v, Y_v), v \in \mathcal{U}_2)$ has the same law as Y. Indeed, recall that the system $((Y_{h(v)}, X_{h(v)}), v \in \mathcal{U}_2)$ defined as in the proof of Lemma 3.3.11 can be viewed as a system generated by the bifurcator (Y, X), then applying Lemma 3.3.12 to this system, we have that $A^Y_{h(\ell^{\mathbb{N}})}$ has the same law as Y. We observe that $\bar{v} = h(\ell^{\mathbb{N}})$ by the construction of h, which entails our claim.*

Lemma 3.3.14. *$\mathbf{W}_{(X,Y)}$ has the same law as $\mathbf{X}^{[\epsilon]}$.*

Proof. We first give some remarks on the process $A^X_{\ell^{\mathbb{N}}}$ defined as in (3.3.8). We know from Lemma 3.3.12 that $A^X_{\ell^{\mathbb{N}}}$ has law P_x, and $A^X_{\ell^{\mathbb{N}}}$ killed at $\zeta_{\ell^{\mathbb{N}}} := \inf \left\{ t \geq 0 : \; A^X_{\ell^{\mathbb{N}}}(t) \leq \epsilon \right\}$ is $A_{\ell^{\mathbb{N}}}$. For every $\ell^n \in \ell^{\mathbb{N}}$ such that $t := \beta_{\ell^n} + \lambda_{\ell^n} \leq \zeta_{\ell^{\mathbb{N}}}$, we have by (3.3.6) that the size of the jump at t is

$$y := -\Delta A^X_{\ell^{\mathbb{N}}}(t) = -\Delta X_{\ell^n}(\lambda_{\ell^n}) = X_{\ell^n r}(0).$$

Note that it is possible that $y > \epsilon$ or $y \leq \epsilon$: if $y \leq \epsilon$, then we know that the particle $\ell^n r$ (together with its progeny) is killed immediately, that is $\lambda_{\ell^n r} = 0$. On the other hand, for all $m \in \mathbb{N}$ and $t' \in (\beta_{\ell^m}, \beta_{\ell^m} + \lambda_{\ell^m})$, we have $-\Delta A^X_{\ell^{\mathbb{N}}}(t') \leq \epsilon$.

Let us now describe the dynamics of $\mathbf{W}_{(X,Y)}$ as the following truncated cell system. The cell system starts with a cell whose size evolves according to $\mathcal{X}_\emptyset := A^X_{\ell^{\mathbb{N}}}$ with law P_x. By killing $\mathcal{X}_\emptyset$ at the time when entering $(0,\epsilon]$, we get $\mathcal{X}^{[\epsilon]}_\emptyset = A_{\ell^{\mathbb{N}}}$. We next build the first generation. The daughter cells in the first generation born (strictly) after $\zeta_{\ell^{\mathbb{N}}}$ are all killed. For each time $t \leq \zeta_{\ell^{\mathbb{N}}}$ with $y := -\Delta\mathcal{X}_\emptyset(t) > \epsilon$, we observe from the remarks above that there exists a certain $w \in \ell^{\mathbb{N}}$ such that $t = \beta_w + \lambda_w$ and $X_{wr}(0) = y$. So a daughter cell is born at t and its size evolves according to $A^X_{wr\ell^{\mathbb{N}}}$, which is the process associated with $wr\ell^{\mathbb{N}} := \{wrv, v \in \ell^{\mathbb{N}}\}$ as in (3.3.8). As $wr\ell^{\mathbb{N}}$ is the left-most branch in the sub-tree $(wrv, v \in \mathcal{U}_2)$, we deduce from Lemma 3.3.12 that $A^X_{wr\ell^{\mathbb{N}}}$ has distribution P_y and $A_{wr\ell^{\mathbb{N}}}$ defined as in (3.3.7) is $A^X_{wr\ell^{\mathbb{N}}}$ killed when entering $(0,\epsilon]$. On the other hand, for every time $t' \leq \zeta_{\ell^{\mathbb{N}}}$ with $y' := -\Delta\mathcal{X}_\emptyset(t') \in (0,\epsilon]$, we agree that the daughter cell born at t' is killed immediately. We hence conclude that those non-degenerate size processes $\mathcal{X}_i^{[\epsilon]}$ with $i \in \mathbb{N}$ are exactly those non-degenerate processes $A_{wr\ell^{\mathbb{N}}}$ with $w \in \ell^{\mathbb{N}}$. The proof is completed by iteration of this argument. $\square$

Proof of Theorem 3.3.9. For every $\epsilon > 0$, applying Lemma 3.3.14 to $\mathbf{W}_{(Y,X)}$, we deduce that $\mathbf{W}_{(Y,X)}$ and $\mathbf{Y}^{[\epsilon]}$ have the same law. Together with Lemma 3.3.11, this implies that $\mathbf{X}^{[\epsilon]}$ and $\mathbf{Y}^{[\epsilon]}$ have the same law. Letting $\epsilon \to 0+$, we conclude by Lemma 3.3.4 that $\mathbf{X}$ and $\mathbf{Y}$ have the same finite-dimensional distributions. $\square$

3.3.4 Proof of Theorem 3.1.1

Using Theorem 3.3.9, we complete the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. The implication $(i) \Rightarrow (ii)$ follows from Lemma 3.2.7 and the equivalence $(iii) \Leftrightarrow (i)$ follows from Corollary 3.2.17. So it remains to prove that $(ii) \Rightarrow (iii)$. Suppose that $X^{(0)}$ and $\tilde{X}^{(0)}$ can be coupled to form a bifurcator. We can check that $X^{(0)}$ and $\tilde{X}^{(0)}$ satisfy **[H]** and **[Hη]**, then we are led to the conclusion that the growth-fragmentations have the same finite dimensional distribution by Theorem 3.3.9. Indeed, fix $q \geq 2$ and $K > \kappa(q)$, then we know from Example 3.3.1 that $X^{(0)}$ satisfies **[H]** with the function $(t,x) \mapsto x^q \mathrm{e}^{-Kt}$; further, it follows from (3.2.3) that $X^{(0)}$ also satisfies **[Hη]** with this function and any $\eta \in (\frac{\kappa(q)-\Phi(q)}{K-\Phi(q)}, 1)$. Similarly, we have that $\tilde{X}^{(0)}$ also satisfies both **[H]** and **[Hη]**. This completes the proof. $\square$

3.3.5 Proof of Theorem 3.1.2

We now turn to self-similar growth-fragmentations. In order to prove Theorem 3.1.2, we first prove the following lemmas.

Lemma 3.3.15. *Let $X^{(\alpha)}$ be a self-similar cell process with index $\alpha \in \mathbb{R}$ related to a SNLP ξ as in (3.1.4). Suppose $\kappa(q) < 0$ for a certain $q > 0$, then $X^{(\alpha)}$ satisfies both* **[H]** *and* **[Hη]** *(for any $\eta \in (1 - \frac{\kappa(q)}{\Phi(q)}, 1)$) with the function $(t,x) \mapsto x^q$.*

Proof. This follows directly from Lemma 2 and Lemma 3 in [20]. $\square$

Lemma 3.3.16. *Let $X^{(\alpha)}$ and $Y^{(\alpha)}$ be two self-similar cell processes with index $\alpha \in \mathbb{R}$ related to SNLPs ξ and γ respectively as in (3.1.4). Suppose that $\kappa = \kappa_\gamma$, then $X^{(\alpha)}$ and $Y^{(\alpha)}$ can be coupled to form a bifurcator.*

Proof. By Proposition 3.2.5 we may assume that ξ is the switching transform of γ with switching time $\tau = \inf\{t \geq 0 : \ \xi(t) \neq \gamma(t)\}$. Say $X^{(\alpha)}(0) = Y^{(\alpha)}(0) = x > 0$, we set

$$\tau^{(\alpha)} := x^{-\alpha} \int_0^\tau \exp(-\alpha \xi(r)) dr,$$

then we have by Lamperti's time-substitution (3.1.4) that $\tau^{(\alpha)} = \inf\{t \geq 0 : X^{(\alpha)}(t) \neq Y^{(\alpha)}(t)\}$ and $X^{(\alpha)}(\tau^{(\alpha)}) + Y^{(\alpha)}(\tau^{(\alpha)}) = X^{(\alpha)}(\tau^{(\alpha)}-)$. Let $\tilde{Y}^{(\alpha)}$ be an independent copy of $Y^{(\alpha)}$ and we build a process

$$\hat{Y}^{(\alpha)}(t) := Y^{(\alpha)}(t)\mathbf{1}_{\{t<\tau^{(\alpha)}\}} + y\tilde{Y}^{(\alpha)}(y^{\alpha}(t-\tau^{(\alpha)}))\mathbf{1}_{\{t\geq\tau^{(\alpha)}\}}, \quad t \geq 0,$$

where $y := -\Delta X^{(\alpha)}(\tau^{(\alpha)})$. Then $(X^{(\alpha)}, \hat{Y}^{(\alpha)})$ is a bifurcator. $\square$

We now complete the proof of Theorem 3.1.2.

Proof of Theorem 3.1.2. $(i) \Rightarrow (ii)$: This follows from Lemma 3.3.16.

$(ii) \Rightarrow (iii)$: Since Lemma 3.3.15 ensures that **[H]** and **[Hη]** hold under assumption (3.1.5), we have from Theorem 3.3.9 that the self-similar growth-fragmentations $\mathbf{X}^{(\alpha)}$ and $\tilde{\mathbf{X}}^{(\tilde{\alpha})}$ have the same finite-dimensional distributions.

$(iii) \Rightarrow (i)$: Suppose that the growth-fragmentations $\mathbf{X}^{(\alpha)}$ and $\tilde{\mathbf{X}}^{(\tilde{\alpha})}$ have the same finite-dimensional distributions. We first know from the self-similarity (Theorem 2 in [20]) that $\alpha = \tilde{\alpha}$.

To prove $\kappa = \tilde{\kappa}$, we first deduce from Proposition 3.3 and its proof in [21] that for every $q > 0$, there is

$$\mathbf{E}_1\left[\int_0^{\infty}\left(\sum_{y\in\mathbf{X}^{(\alpha)}(t)} y^{q+\alpha}\right)dt\right] = \begin{cases} -\frac{1}{\kappa(q)}, & \text{if } \kappa(q) < 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.3.10}$$

Note from Corollary 4 in [20] that the integrand possesses càdlàg paths under assumption (3.1.5). As $\mathbf{X}^{(\alpha)}$ and $\tilde{\mathbf{X}}^{(\alpha)}$ have the same same finite-dimensional distributions, we thus deduce that for every $q > 0$ with $\kappa(q) < 0$, there is $\tilde{\kappa}(q) = \kappa(q) < 0$. Therefore, if there exists $q_0 > 0$ such that $\kappa(q) < 0$ for all $q > q_0$, then $\kappa(q) = \tilde{\kappa}(q)$ for all $q > q_0$. Otherwise, by the convexity of κ there exists $\omega > 0$, which is the largest root of κ, such that $\kappa(q) > 0$ for all $q > \omega$. It follows from (3.3.10) that ω is also the largest root for $\tilde{\kappa}$ and $\tilde{\kappa}(q) > 0$ for all $q > \omega$. We hence deduce from Theorem 3.5 in [21] that $\tilde{\kappa}(q) = \kappa(q)$ for all $q > \omega$. Summarizing the two cases, we conclude that there exists a certain constant $a > 0$ such that $\kappa(q) = \tilde{\kappa}(q)$ for all $q > a$, which entails that $\kappa = \tilde{\kappa}$. $\square$

3.3.6 A self-similar growth-fragmentation connected with random planar maps

We end this section by an illustration of Theorem 3.1.2. For $p \geq 1$, a *Boltzmann triangulation of the p-gon* is chosen at random according to a Boltzmann type distribution from the set of all the different random maps whose faces are all triangles except the external face, which is a polygon with p vertices. For every $r \geq 0$, consider all the vertices whose distances to the boundary (of the external face) is r and the edges linking them, this gives rise to a collection of cycles. A recent work [22] shows that as $p \to \infty$, the process indexed by the distance r of the multiset of appropriately rescaled lengths of these cycles converges in distribution to a self-similar growth-fragmentation process with index $\alpha = -\frac{1}{2}$ associated with a SNLP ξ_B characterized by $(0, -\frac{8}{3}, \Lambda_B, 0)$, where

$$\Lambda_B(dz) = \mathrm{e}^{-3z/2}(1-\mathrm{e}^z)^{-5/2}\mathbf{1}_{\{-\log 2 < z < 0\}}dz.$$

Thus the cumulant of ξ_B defined as in (3.1.3) is

$$\kappa_B(q) = \frac{4\sqrt{\pi}}{3}\frac{\Gamma(q-\frac{3}{2})}{\Gamma(q-3)}, \quad q \geq 2,$$

where Γ is the gamma function. Noticing that $\kappa_B(q) < 0$ for $2 < q < 3$, we deduce from Theorem 3.1.2 that this self-similar growth-fragmentation has characteristics $(\kappa_B, -\frac{1}{2})$. In particular, for every $a > 3/2$ let X_a be a self-similar cell process of index $-\frac{1}{2}$ associated by (3.1.4) with a SNLP with characteristics $(0, c_a, \Lambda_a, 0)$, where

$$c_a = -\frac{8}{3} + \int_{1/2}^{1}(1-2\mathrm{e}^z)(1-\mathrm{e}^z)^a\mathrm{e}^{-3z/2}(1-\mathrm{e}^z)^{-5/2}dz,$$

$$\Lambda_a(dz) = \left((1-(1-\mathrm{e}^z)^a)\mathbf{1}_{\{-\log 2 < z < 0\}} + \mathrm{e}^{az}\mathbf{1}_{\{z < -\log 2\}}\right)\mathrm{e}^{-3z/2}(1-\mathrm{e}^z)^{-5/2}dz,$$

then the growth-fragmentation driven by X_a is also a self-similar growth-fragmentation with characteristics $(\kappa_B, -\frac{1}{2})$.

Chapter 4

Ornstein-Uhlenbeck type Growth-fragmentation processes

This chapter is mainly based on a work in preparation [77].

Growth-fragmentation processes describe the evolution of particles that grow and divide as time proceeds. Previous studies on growth-fragmentations have mostly focused on the self-similar case. We introduce a new type of growth-fragmentations which are closely related to Lévy driven Ornstein-Uhlenbeck type processes. Our main results show that such growth-fragmentations fulfill a law of large numbers under certain conditions. This model is partly motivated by a study by Baur and Bertoin [10] of the destruction process of an infinite recursive tree, where a certain Ornstein-Uhlenbeck type growth-fragmentation naturally arises.

4.1 Introduction

Fragmentation processes describe particles that split randomly as time passes, independently one of the others. See [17] for a comprehensive overview. Recently, Bertoin [20, 19] extended fragmentations to *growth-fragmentation processes*, in which a particle may also grow and decay continuously. In both (pure) fragmentation and growth-fragmentation, research has been focused on the *self-similar* case, which means the particle system behaves the same when viewed at certain different scales on space and time.

In the present work, we propose a new type of growth-fragmentation processes that possess a different scaling property. We name them *Ornstein-Uhlenbeck (OU) type growth-fragmentation processes*, as in such a particle system, informally speaking, each particle splits and grows independently, whose size evolves according to the exponential of a certain *OU type process* $(Z(t), t \geq 0)$ driven by a Lévy process ξ:

$$Z(t) := \mathrm{e}^{-\theta t} Z(0) + \int_0^t \mathrm{e}^{-\theta(t-s)} d\xi(s), \quad t \geq 0, \tag{4.1.1}$$

where $\theta \in \mathbb{R}$ and the integral is defined in the sense of a stochastic integral, as the Lévy process ξ is a semimartingale. If ξ is a Brownian motion, then Z is a well-known Gaussian OU process.

Our model is initially motivated by a recent work [10] (see also a related work [67]), results in which imply that a certain OU type growth-fragmentation naturally arises in dynamical percolation on an infinite recursive tree (see Section 4.5). Besides this motivation, our model may have potential applications, as OU type processes are widely applied in various domains: in biology, they are used in a neuronal model with signal-dependent noise [60]; in finance, they are used in an option price model with stochastic volatility [7, 8], to name just a few.

We now give a more precise description of OU type growth-fragmentations. Let $c_0^{\downarrow}$ be the space of decreasing null sequences (that converge to 0), endowed with the ℓ^{∞}-norm. An OU type growth-fragmentation process is a $c_0^{\downarrow}$-valued càdlàg Markov process

$$\mathbf{X}^{\downarrow} = \Big(\mathbf{X}^{\downarrow}(t) := (X_1(t), X_2(t), \ldots),\ t \geq 0\Big),$$

where $\mathbf{X}^{\downarrow}(t)$ is viewed as the decreasing sequence of the size of the particles alive at time t. For every $x \in [0, \infty)$, let $\mathbf{P}_x$ denote for the law of $\mathbf{X}^{\downarrow}$ with initial value $\mathbf{X}^{\downarrow}(0) = (x, 0, \ldots) \in c_0^{\downarrow}$. The process $\mathbf{X}^{\downarrow}$ further satisfies the following properties:

(P1) (The branching property) For a sequence $\mathbf{x} = (x_1, x_2, \ldots) \in c_0^{\downarrow}$, and every $t \geq 0$, the distribution of $\mathbf{X}^{\downarrow}$ given $\mathbf{X}^{\downarrow}(0) = \mathbf{x}$ is the same as the union of the masses, arranged in the decreasing order, of a sequence of independent fragmentations $(\mathbf{X}^{[i]\downarrow})_{i\geq 1}$, where each $\mathbf{X}^{[i]\downarrow}$ has distribution $\mathbf{P}_{x_i}$.

(P2) (The OU property) There exists an certain index $\theta \in \mathbb{R}$, such that for every $x \in [0, \infty)$, the distribution of the rescaled process $(x^{\exp(-\theta t)}\mathbf{X}^{\downarrow}(t))_{t\geq 0}$ under $\mathbf{P}_1$ is $\mathbf{P}_x$.

The branching property means that the fragments evolve independently. The OU property is an analogue of the scaling property of the exponential of an OU type process (a direct consequence of (4.1.1)), which signifies that the mass of each fragment grows or decays gradually towards (resp. away from) an equilibrium value when $\theta > 0$ (resp. $\theta < 0$). For comparison, we recall that a *self-similar growth-fragmentation* $\mathbf{X}^{\downarrow}$ (in particular it can be a self-similar fragmentation) fulfills the same branching property, but a different scaling property, namely, for a certain index $\alpha \in \mathbb{R}$, the rescaled process $(x\mathbf{X}^{\downarrow}(x^{\alpha}t))_{t\geq 0}$ under $\mathbf{P}_1$ is $\mathbf{P}_x$. See Theorem 2 in [20] and Definition 2 in [11]. Note that, however, the OU type scaling property does not have an analogue in (pure) fragmentations.

Hence the first main object of this work is to provide a construction of such processes. The starting point of our approach is the observation that upon a logarithmic transformation, the discrete time skeletons of homogeneous (self-similar with index $\alpha = 0$) fragmentations can be viewed as branching random walks [25]. If we extend branching random walks to general branching systems, in which an atom may also move continuously, then we naturally obtain growth-fragmentations by an exponential transform. Bertoin [19] introduced this idea and developed a construction of homogeneous growth-fragmentations (which he called *compensated fragmentation processes*) by associating with *branching Lévy processes*. It is remarkable that a truncation procedure is also used in this approach such that the branching events are allowed to occur with an infinite intensity. See also [26] for a related construction of binary self-similar growth-fragmentations.

Similarly, we construct certain *branching OU type processes* which naturally yields OU type growth-fragmentations in the sense that they fulfill the desired properties listed above. We stress that in our model the branching rate could also be infinite. The technical difficulty in adopting this approach is that one needs to check that this growth-fragmentation does not explode, that is, for every $x > 0$, only a finite number of fragments have size greater than x at any time.

OU type growth-fragmentations have interesting behaviors that are crucially different from the well-known self-similar case. Roughly speaking, under certain conditions the size of a "typical" fragment in an OU type growth-fragmentation evolves as the exponential of a certain OU type process and converges in distribution to a stationary distribution (see Lemma 4.2.2), and two typical fragments are "almost independent", we hence prove a law of large numbers. This result should be compared with the limit theorems for empirical measures of self-similar fragmentations and growth-fragmentations [23, 38], as well as the law of large numbers in the context of branching Gaussian OU processes [43]. We also find that OU type growth-fragmentations bear a connection with Bertoin's *Markovian growth-fragmentations* [20] and that they are the stochastic counterparts of certain (deterministic) growth-fragmentation equations, see [30, 31, 39, 46] for related works on the latter topic.

The rest of this paper is organized as follows. In Section 4.2 we construct OU type growth-fragmentation processes and establish some important properties. We continue to investigate OU type growth-fragmentations in Section 4.3. Specifically, we build a relation to certain growth-fragmentation equations, and prove a law of large numbers. In Section 4.4 we discuss the connections between OU type growth-fragmentations and Bertoin's Markovian growth-fragmentations [20]. Finally, we describe an OU type growth-fragmentation related to a destruction of the infinite recursive tree in Section 4.5.

4.2 Construction of OU type growth-fragmentation processes

In this section we present the construction of OU type growth-fragmentation processes. We first recall some background on OU type processes and a connection between homogeneous fragmentations and branching random walks. Then we construct branching OU type processes. Using the latter, we introduce OU type growth-fragmentation processes and establish some fundamental properties.

4.2.1 Preliminaries: Ornstein-Uhlenbeck type processes

Let us present some fundamental background on Ornstein-Uhlenbeck (OU) type processes driven by Lévy processes, see [5] or Section 17 in [73]. We also refer to [12] for properties of Lévy processes. Implicitly, throughout this work we only consider OU type processes without positive jumps.

Let ξ be a Lévy process with no positive jumps, possibly killed, which is often referred to as a **spectrally negative Lévy process (SNLP)**. The SNLP ξ is characterized by its Laplace exponent $\Phi : [0, \infty) \to \mathbb{R}$ such that

$$\mathbb{E}\left[e^{q\xi(t)}\right] = e^{\Phi(q)t}, \qquad \text{for all } t, q \geq 0.$$

The function Φ is convex and is given by the Lévy-Khintchine formula

$$\Phi(q) = -k + \frac{1}{2}\sigma^2 q^2 + cq + \int_{(-\infty,0)} \left(e^{qz} - 1 + q(1 - e^z)\right) \Lambda(dz), \quad q \geq 0, \tag{4.2.1}$$

where $k \geq 0$, $\sigma \geq 0$, $c \in \mathbb{R}$, and the Lévy measure Λ on $(-\infty, 0)$ satisfies

$$\int_{(-\infty,0)} (|z|^2 \wedge 1)\Lambda(dz) < \infty. \tag{4.2.2}$$

We call ξ a SNLP with characteristics (σ, c, Λ, k). In the Lévy-Khintchine formula, we can also replace $q(1-e^z)$ in the integral by $-qz\mathbf{1}_{\{z>-1\}}$, as often in the literature, then we need to change the drift coefficient c.

Let $\theta \in \mathbb{R}$ be a real constant and ξ be a SNLP with characteristics $(\sigma^2, c, \Lambda, k)$, we define an **Ornstein-Uhlenbeck (OU) type process** Z with characteristics $(\sigma, c, \Lambda, k, \theta)$ or simply (Φ, θ), starting from $Z(0) = z \in \mathbb{R}$, by

$$Z(t) = e^{-\theta t} z + \int_0^t e^{-\theta(t-s)} d\xi(s), \quad t \geq 0, \tag{4.2.3}$$

where the integral is defined in the sense of a stochastic integral, as the Lévy process ξ is a semimartingale. By convention, if ξ is killed at time $\zeta \geq 0$, then $Z(t) := -\infty$ for every $t \geq \zeta$. It is well-known that Z is the path-wise unique solution (see (17.2) in [73]) of the stochastic integral equation

$$Z(t) = z + \xi(t) - \theta \int_0^t Z(s)ds.$$

When $\theta > 0$, Z is often called an *inward* OU type process; respectively, while $\theta < 0$, Z is called an *outward* OU type process. Note that in the literature, classical OU type processes often only refer to the inward case ($\theta > 0$).

The next observation plainly follows from (4.2.3).

Lemma 4.2.1. *If Z_1 and Z_2 are independent OU type processes with respective characteristics (Φ_1, θ) and (Φ_2, θ), then $Z_1 + Z_2$ is an OU type process with characteristics $(\Phi_1 + \Phi_2, \theta)$.*

Denote by P_z the law of the OU type process Z starting from $Z(0) = z$, then we observe from (4.2.3) that

$$\text{the process } (\mathrm{e}^{-\theta t} z + Z(t))_{t\geq 0} \text{ under } P_0 \text{ has law } P_z. \tag{4.2.4}$$

Write E_z for the mathematical expectation under P_z, then we have for every $t \geq 0$ that

$$E_z[\exp(qZ(t))] = \exp\left(\mathrm{e}^{-\theta t} zq + \int_0^t \Phi(q\mathrm{e}^{-\theta s})ds\right), \quad q \geq 0. \tag{4.2.5}$$

Under certain conditions, an inward OU type process converges in distribution to its stationary distribution.

Lemma 4.2.2 (Theorem 17.5 and 17.11 in [73]). *If $\theta > 0$ and Λ satisfies*

$$\int_{(-\infty, -\log 2)} \log|z| \ \Lambda(dz) < \infty, \tag{4.2.6}$$

then the OU type process Z possesses a unique stationary distribution Π, which is a probability measure on $\mathbb{R}$ with Laplace transform

$$\int_{\mathbb{R}} \mathrm{e}^{qy} \Pi(dy) = \exp\left(\int_0^\infty \Phi(\mathrm{e}^{-\theta s} q) ds\right), \quad q \geq 0.$$

Further, for every $z \in \mathbb{R}$ and bounded and continuous function $g : \mathbb{R} \to \mathbb{R}$ there is

$$\lim_{t\to\infty} \mathbb{E}_z\left[g(Z(t))\right] = \int_{\mathbb{R}} g(y)\Pi(dy).$$

If (4.2.6) does not hold, then Z does not have any stationary distribution.

We remark that the stationary distribution Π is *self-decomposable*, which means that if a random variable Y has law Π, then for every constant $r \in (0,1)$, there exists an independent random variable $Y^{(r)}$, such that $Y \overset{d}{=} rY + Y^{(r)}$. Conversely, every self-decomposable measure is the stationary distribution of a certain OU type process. See Definition 15.1 and Theorem 17.5 in [73] for details.

4.2.2 Homogeneous fragmentation processes

We present in this section a connection between homogeneous fragmentations and branching random walks, which was developed in Section 2 in [19]. This will help us to understand the construction of OU type growth-fragmentations.

Let ν be a **finite** measure on the space of mass-partitions denoted by

$$\mathcal{S} := \left\{ \mathbf{s} := (s_1, s_2, \ldots) : \ s_1 \geq s_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} s_i \leq 1 \right\}.$$

A homogeneous fragmentation $\mathbf{X}^{\downarrow}$ with no erosion and finite dislocation measure ν describes the following particle system. Initially, there is a single particle with mass 1. Each particle of mass $x > 0$ splits at rate $\nu(\mathcal{S})$, and generates a sequence of particles with masses $(xs_i, i \in \mathbb{N})$, where $(s_1, s_2, \ldots) \in \mathcal{S}$ has distribution $\nu(\cdot)/\nu(\mathcal{S})$. Each child fragment continues in a similar way. Upon a logarithm transform, this is a (continuous time) branching random walk.

Let us introduce some notation and give a formal construction of $\mathbf{X}^{\downarrow}$ via this branching random walk. We index the fragments by the *Ulam-Harris tree* $\mathcal{U} := \bigcup_{n=0}^{\infty} \mathbb{N}^n$ with $\mathbb{N}^0 = \{\emptyset\}$ by convention, so an element $u \in \mathcal{U}$ is a finite sequence of natural numbers $u = (n_1, \ldots, n_{|u|})$ where $|u| \in \mathbb{N}$ stands for the generation of u. Write $u_- = (n_1, \ldots, n_{|u|-1})$ for her mother and $uk = (n_1, \ldots n_{|u|}, k)$ for her k-th daughter with $k \in \mathbb{N}$. We also introduce the space

$$\mathcal{R} := \left\{ \mathbf{r} = (r_1, r_2, \ldots) \ : \ 0 \geq r_1 \geq r_2 \geq \ldots \geq -\infty, \sum_{i=1}^{\infty} \mathrm{e}^{r_i} \leq 1 \right\},$$

and a finite measure μ on $\mathcal{R}$, which is the image of ν by the map from $\mathbf{s} \in \mathcal{S}$ to $\log \mathbf{s} \in \mathcal{R}$ (with convention $\log 0 := -\infty$).

Proposition 4.2.3 ([19]). *Suppose that $0 < \mu(\mathcal{R}) < \infty$. Let $(\lambda_u, u \in \mathcal{U})$ be a family of i.i.d. exponential random variables with parameter $\mu(\mathcal{R})$, $((\Delta a_{ui})_{i \in \mathbb{N}}, u \in \mathcal{U})$ be a family of i.i.d. random variables with distribution $\mu(\cdot)/\mu(\mathcal{R})$. The two families are independent. With initial values $b_{\emptyset} = 0$ and $a_{\emptyset} = 0$, we define recursively*

$$a_{ui} = a_u + \Delta a_{ui}, \quad b_{ui} = b_u + \lambda_u, \qquad \textit{for every } u \in \mathcal{U}, i \in \mathbb{N}.$$

For every $u \in \mathcal{U}$ the triple (a_u, b_u, λ_u) stands for the position, the birth time and the lifetime respectively of the particle indexed by u. For every $t \geq 0$, the multiset (which is like a set but allows multiple instances of elements)

$$\mathbf{Z}(t) := \{\!\{ a_u \in \mathbb{R} : \ u \in \mathcal{U}, t \in [b_u, b_u + \lambda_u) \}\!\}$$

by the positions of particles alive at time t. Let $\mathbf{X}^{\downarrow}(t) := (X_1(t), X_2(t), \ldots)$ be the null-sequence obtained by listing the element of $\mathbf{X}(t) := \{\!\{ \exp(z), \ z \in \mathbf{Z}(t) \}\!\}$ in decreasing order. Then the process $\mathbf{X}^{\downarrow}$ is a homogeneous fragmentation with no erosion and finite dislocation measure ν.

Note that if $\Delta a_{ui} = -\infty$, then by convention $a_{ui} := -\infty$, which means that the atom ui (as well as its descendants) is not taken into account.

We now introduce a different representation of $\mathbf{Z}$. The key point is that we now distinguish between two types of branching events, namely, those in which exactly one particle is generated, which corresponds to those $u \in \mathcal{U}$ such that $(\Delta a_{u1}, \Delta a_{u2}, \ldots)$ is included in

$$\mathcal{R}_1 := \{\mathbf{r} \in \mathcal{R} : \; r_1 > \infty, r_2 = r_3 = \ldots = -\infty\},$$

and the others (those correspond to $\mathcal{R} \setminus \mathcal{R}_1$). We shall next treat the former as displacements of atoms, but not as branching events, and thus changes accordingly the genealogy of this branching random walk. From this point of view, we have the following description.

Proposition 4.2.4 ([19]). *Suppose that $0 < \mu(\mathcal{R}) < \infty$. Let $(\lambda_u, u \in \mathcal{U})$ be a family of i.i.d. exponential random variables with parameter $\mu(\mathcal{R} \setminus \mathcal{R}_1)$, $((\Delta a_{ui})_{i \in \mathbb{N}}, u \in \mathcal{U})$ be a family of i.i.d. random variables with distribution $\mu(\cdot)/\mu(\mathcal{R} \setminus \mathcal{R}_1)$, and $(\xi_u, u \in \mathcal{U})$ be a family of i.i.d. compound Poisson processes with Lévy measure given by the image of the restriction of $\mu|_{\mathcal{R}_1}$ via the map $\mathbf{r} \to r_1$ from $\mathcal{R}_1$ to $(-\infty, 0)$. With initial values $b_\emptyset = 0$ and $a_\emptyset = 0$, we define recursively*

$$a_{ui} = a_u + \Delta a_{ui} + \xi_u(\lambda_u), \quad b_{ui} = b_u + \lambda_u, \qquad \text{for every } u \in \mathcal{U}, i \in \mathbb{N}.$$

For every $u \in \mathcal{U}$ the triple (a_u, b_u, λ_u) stands for the position, the birth time and the lifetime respectively of the particle indexed by u. For every $t \geq 0$, we define the multiset

$$\tilde{\mathbf{Z}}(t) := \{\!\{ a_u + \xi_u(t - b_u) : \; u \in \mathcal{U}, t \in [b_u, b_u + \lambda_u) \}\!\}$$

by the positions of particles alive at time t, and let $\tilde{\mathbf{X}}^{\downarrow}(t) := (X_1(t), X_2(t), \ldots)$ be the null-sequence obtained by listing the elements of $\{\!\{\exp(z), \; z \in \tilde{\mathbf{Z}}(t)\}\!\}$ in decreasing order. Then the process $\tilde{\mathbf{X}}^{\downarrow}$ is a homogeneous fragmentation with no erosion and finite dislocation measure ν.

Recall that the *selected fragment* of the homogeneous fragmentation $\mathbf{X}^{\downarrow}$ is obtained as follows. At the first dislocation of $\mathbf{X}^{\downarrow}$, we keep the largest child fragment and discard all the others. Next we consider the first dislocation of this only fragment that remains, and select its largest child. We continue so on and so forth. Therefore, for each time $t \geq 0$ there is only one selected fragment, whose size is denoted by $X_*(t)$. We can express the selected fragment by using the system $\tilde{Z}$. Let $\partial\mathcal{U}$ denote the set of infinite sequences of positive integers. Consider the *oldest* branch $\bar{1} = (1, 1, 1, \ldots) \in \partial\mathcal{U}$, then clearly

$$X_*(t) = \exp\left(a_{1^n} + \xi_{1^n}(t - b_{1^n})\right), \qquad t \in [b_{1^n}, b_{1^n} + \lambda_{1^n}), \tag{4.2.7}$$

where $1^n := (1, 1, \ldots, 1) \in \mathbb{N}^n$ for every $n \geq 0$. Then $\log X_*$ is a compound Poisson process with Laplace exponent

$$\int_{\mathcal{R}} (\mathrm{e}^{q r_1} - 1)\mu(d\mathbf{r}), \qquad q \geq 0.$$

See Lemma 1 in [19]. We stress that the law of $\log X_*$ is different from ξ.

4.2.3 OU type branching Markov chain

Following the discussion in the introduction, we now extend the construction of $\tilde{\mathbf{Z}}$ as in Proposition 4.2.4 to *OU type branching Markov chains*, then intuitively we derive OU type growth-fragmentations from such systems by an exponential transform.

Recall that in Proposition 4.2.4, the movement of an atom is a compound Poisson process ξ with Lévy measure Λ_*, which is the image of the restriction of $\mu|_{\mathcal{R}_1}$ via the map $\mathbf{r} \to r_1$ from $\mathcal{R}_1$ to $(-\infty, 0)$. Here we replace ξ by an OU type process Z with characteristics $(\sigma, c, \Lambda_*, 0, \theta)$, where $\sigma^2 \geq 0$, $c \in \mathbb{R}$ and $\theta \in \mathbb{R}$. The splitting mechanism is still given by $\mu|_{\mathcal{R}\setminus\mathcal{R}_1}$ such that a particle at position y splits into two or more particles at $y + \mathbf{r}$ with rate $\mu|_{\mathcal{R}\setminus\mathcal{R}_1}(d\mathbf{r})$; the particle born at position $y + r_i$ evolves according to the law of Z with $Z(0) = y + r_i$.

Further, we observe that to define these dynamics, we do not need μ to be finite. It suffices to suppose that μ is a sigma-finite measure on $\mathcal{R}$ that satisfies

$$\int_{\mathcal{R}} (1 - \mathrm{e}^{r_1})^2 \mu(d\mathbf{r}) < \infty, \tag{4.2.8}$$

and that $\mu(\mathcal{R} \setminus \mathcal{R}_1) < \infty$. Then Λ_* is still a Lévy measure (that satisfies (4.2.2)) such that Z is well-defined, and this particle system (with a different drift coefficient) can be rigorously constructed in the following way.

Definition 4.2.5. *Let $\theta \in \mathbb{R}$, $\sigma^2 \geq 0$, $c \in \mathbb{R}$, and μ be a sigma-finite measure in $\mathcal{R}$ such that (4.2.8) holds and $\mu(\mathcal{R} \setminus \mathcal{R}_1) < \infty$. Consider three independent processes $(\lambda_u)_{u\in\mathcal{U}}$, $(\xi_u)_{u\in\mathcal{U}}$ and $(\Delta a_{ui}, i \in \mathbb{N})_{u\in\mathcal{U}}$:*

- *$(\lambda_u)_{u\in\mathcal{U}}$ is a family of i.i.d. exponential variables with parameter $\mu(\mathcal{R}\setminus\mathcal{R}_1)$.*
- *$(Z_u)_{u\in\mathcal{U}}$ is a family of i.i.d. OU type processes starting from $Z_u(0) = 0$ with characteristics (ψ, θ), where*

$$\psi(q) := \frac{1}{2}\sigma^2 q^2 + \Big(c + \int_{\mathcal{R}\setminus\mathcal{R}_1} (1 - \mathrm{e}^{r_1})\mu(d\mathbf{r})\Big) q + \int_{\mathcal{R}_1} \left(\mathrm{e}^{q r_1} - 1 + q(1 - \mathrm{e}^{r_1})\right) \mu(d\mathbf{r}). \tag{4.2.9}$$

- *$(\Delta a_{ui}, i \in \mathbb{N})_{u\in\mathcal{U}}$ is a family of i.i.d. sequences, each sequence being distributed according to the conditional probability $\mu(\cdot \mid \mathcal{R} \setminus \mathcal{R}_1)$.*

Set $b_\emptyset = 0$ and $a_\emptyset = 0$. Write $b_u := \sum_{j=0}^{|u|-1} \lambda_{(u_1,\dots,u_j)}$ for the birth time, $a_{ui} := \mathrm{e}^{-\theta\lambda_u} a_u + Z_u(\lambda_u) + \Delta a_{ui}$ for the position at birth, and $(\mathrm{e}^{-\theta r} a_u + Z_u(r))_{r\geq 0}$ for the movement, which has the law of Z with $Z(0) = a_u$ by (4.2.4). Then the positions of the particles alive at time $t \geq 0$ form a multiset

$$\mathbf{Z}(t) := \{\!\{\mathrm{e}^{-\theta(t-b_u)} a_u + Z_u(t - b_u) : \ u \in \mathcal{U}, b_u \leq t < b_u + \lambda_u\}\!\}, \quad t \geq 0.$$

The process $\mathbf{Z}$ is called an ***OU type branching Markov chain*** *with characteristics (σ, c, μ, θ).*

The choice of the drift coefficient in (4.2.9) is for the following purposes. First, this is consistent with Definition 1 in [19], so for the case $\theta = 0$, an OU type branching Markov chain with characteristics $(\sigma^2, c, \mu, 0)$ is a *branching Lévy process* with characteristics (σ^2, c, μ); second, this will make notation simpler in future use (especially in Lemma 4.2.7); third, this is coherent with the selected atom as the following statement shows.

Lemma 4.2.6. *The selected atom Z_* that corresponds to $\bar{1}$ (in a similar way as (4.2.7)) is an OU type process with characteristics $(\sigma, c, \Lambda_*, 0, \theta)$, where Λ_* is the image of μ via the map $\mathbf{r} \to r_1$ from $\mathcal{R}$ to $(-\infty, 0)$. Equivalently, Z_* has characteristics (Φ_*, θ), where*

$$\Phi_*(q) = \frac{1}{2}\sigma^2 q^2 + cq + \int_{\mathcal{R}} \left(\mathrm{e}^{qr_1} - 1 + q(1 - \mathrm{e}^{r_1})\right) \mu(d\mathbf{r}), \quad q \geq 0.$$

For future use, we present an embedding property that is similar to Lemma 3 in [19]. For each $\ell \geq 0$, we cut an OU type branching Markov chain $\mathbf{Z}$ with characteristics (σ, c, μ, θ) at level ℓ, by keeping at each dislocation the child particle which is the closest to the parent, and by suppressing the other child particles if and only if its distance to the position of the parent at death is larger than or equal to ℓ. Let $B(\ell) \subset \mathcal{U}$ be the set of individuals that are killed by this cutting operation, so $u = (u_1, \ldots, u_{|u|}) \in B(\ell)$ if and only if

$$\Delta a_{u_1,\ldots,u_j} \leq -\ell \text{ and } u_j \geq 2 \text{ for some } j = 1, \ldots, |u|.$$

For every $r \in [-\infty, 0]$, set

$$r^{(\ell)} := \begin{cases} r & \text{if } r > -\ell, \\ -\infty & \text{otherwise.} \end{cases}$$

Then for every $\mathbf{r} = (r_1, r_2, r_3, \ldots) \in \mathcal{R}$, we define

$$\mathbf{r}^{(\ell)} := (r_1, r_2^{(\ell)}, r_3^{(\ell)}, \ldots). \tag{4.2.10}$$

Let $\mu^{(\ell)}$ be the image of μ by the map $\mathbf{r} \mapsto \mathbf{r}^{(\ell)}$.

Lemma 4.2.7. *The truncated process*

$$\mathbf{Z}^{(\ell)}(t) := \{\!\{\mathrm{e}^{-\theta(t-b_u)} a_u + Z_u(t - b_u) : \ u \in \mathcal{U}, u \notin B(\ell), b_u \leq t < b_u + \lambda_u\}\!\}, \quad t \geq 0$$

is an OU type branching Markov chain with characteristics $(\sigma, c, \mu^{(\ell)}, \theta)$.

The proof is deferred to Section 4.2.6.

4.2.4 OU type branching Markov processes

In this section we extend OU type branching Markov chains to a more general class of *OU type branching Markov processes*. Specifically, we release the assumption that $\mu(\mathcal{R} \setminus \mathcal{R}_1) < \infty$ but only suppose that (4.2.8) holds, such that the branching rate could be infinite. We shall use the approach of Definition 2 in [19], which relies on the key embedding property that enables us to consider increasing limits.

Specifically, for every $\ell \geq 0$, write $\mu^{(\ell)}$ for the image of μ by the map $\mathbf{r} \mapsto \mathbf{r}^{(\ell)}$. Then we have for every $\ell \geq 0$ that

$$\mu^{(\ell)}(\mathcal{R} \setminus \mathcal{R}_1) = \mu(\mathbf{r}^{(\ell)} \notin \mathcal{R}_1) = \mu(r_1 = -\infty \text{ or } r_2 > -\ell) \leq \mu(1 - \mathrm{e}^{r_1} > \mathrm{e}^{-\ell})) < \infty.$$

By Lemma 4.2.7 and Kolmogorov's extension theorem, we can build a family of processes on the same probability space, which we still denote by $(\mathbf{Z}^{(\ell)})_{\ell \geq 0}$, such that each $\mathbf{Z}^{(\ell)}$ is an OU type branching Markov chain with characteristics $(\sigma, c, \mu^{(\ell)}, \theta)$, and

$$(\mathbf{Z}^{(\ell)})^{(\ell')} = \mathbf{Z}^{(\ell')} \quad \text{for every } \ell' > \ell,$$

where $(\mathbf{Z}^{(\ell)})^{(\ell')}$ denotes the process obtained by cutting $\mathbf{Z}^{(\ell)}$ at level ℓ'.

Definition 4.2.8. *Suppose that* (4.2.8) *holds. In the notation above, we define (by the increasing limit)*

$$\mathbf{Z}(t) := \lim_{\ell \to \infty} \uparrow \mathbf{Z}^{(\ell)}(t), \quad t \geq 0.$$

We call $\mathbf{Z}$ *an* **OU type branching (Markov) process** *with characteristics* (σ, c, μ, θ).

4.2.5 OU type growth-fragmentation processes

We finally construct OU type growth-fragmentation processes.

Let $\sigma \geq 0$, $c \in \mathbb{R}$, $\theta \in \mathbb{R}$ and ν be a sigma-finite measure on the space of mass-partitions $\mathcal{S}$. We further suppose that ν satisfies

$$\int_{\mathcal{S}} (1 - s_1)^2 \nu(d\mathbf{s}) < \infty. \tag{4.2.11}$$

Write μ for the image of measure ν by the map $\mathbf{s} \mapsto (\log(s_1), \log(s_2), \ldots) \in \mathcal{R}$, then μ satisfies (4.2.8). Let us construct by Definition 4.2.8 an OU type branching Markov process $\mathbf{Z}$ with characteristics (σ, c, μ, θ). Then

$$\mathbf{X}(t) := \{\!\{\exp(z) : \ z \in \mathbf{Z}(t)\}\!\}, \quad t \geq 0$$

can be naturally viewed as a growth-fragmentation process. The crucial property that we need to prove is that $\mathbf{X}$ does not explode, i.e. for every $t, a \geq 0$, $\mathbf{X}(t)$ has a finite number of elements in $[a, \infty)$. In this direction, we introduce the *cumulant* $\kappa : [0, \infty) \to (-\infty, \infty]$:

$$\kappa(q) = \frac{1}{2}\sigma^2 q^2 + cq + \int_{\mathcal{S}} \left(\sum_{i=1}^{\infty} s_i^q - 1 + q(1 - s_1) \right) \nu(d\mathbf{s}), \quad q \geq 0, \tag{4.2.12}$$

with convention $0^0 := 0$. The cumulant κ plays an important role in this work (and also for compensated fragmentations [19]). We denote

$$\mathrm{dom}(\kappa) := \{q \geq 0 : \ \kappa(q) < \infty\}.$$

Since $s_1^q - 1 + q(1-s_1) = \mathcal{O}(1-s_1)^2$, we see from (4.2.11) that

$$q \in \mathrm{dom}(\kappa) \text{ if and only if } \quad \int_{\mathcal{S}} \sum_{i=2}^{\infty} s_i^q \ \nu(d\mathbf{s}) < \infty.$$

As $\sum_{i=2}^{\infty} s_i^q \leq (1-s_1)^q$, we deduce from (4.2.11) that $[2,\infty) \subset \mathrm{dom}(\kappa)$. Note that κ is convex. Recall that $c_o^{\downarrow}$ is the space of all decreasing null sequences endowed with the ℓ^{∞}-distance, i.e. $\|\mathbf{x}^{\downarrow}\|_{\infty} = \sup_{i\in\mathbb{N}} |x_i|$ for $\mathbf{x}^{\downarrow} = (x_1, x_2, \dots) \in c_o^{\downarrow}$.

Theorem 4.2.9. *For every $t \geq 0$, the elements of $\mathbf{X}(t)$ can be rearranged in decreasing order, which yields a decreasing null sequence*

$$\mathbf{X}^{\downarrow}(t) := (X_1(t), X_2(t), \dots) \in c_o^{\downarrow}.$$

Further, for every $\alpha \in \mathrm{dom}(\kappa)$ and $q \geq \alpha(1 \vee \mathrm{e}^{\theta t})$, we have

$$\mathbb{E}\left[\sum_{i=1}^{\infty} X_i(t)^q\right] = \exp\left(\int_0^t \kappa(q\mathrm{e}^{-\theta s}) ds\right). \tag{4.2.13}$$

The proof of Theorem 4.2.9 is postponed to Section 4.2.6.

Definition 4.2.10. *In notation of Theorem 4.2.9, the process $\mathbf{X}^{\downarrow} := (\mathbf{X}^{\downarrow}(t), t \geq 0)$ is called an* **OU type growth-fragmentation process** *with characteristics (σ, c, ν, θ).*

Remark 4.2.11. *When $\theta = 0$, an OU type growth-fragmentation with characteristics $(\sigma, c, \nu, 0)$ is a compensated fragmentation with characteristics (σ, c, ν) in the sense of Definition 3 in [19]. To avoid duplication of existing literature, this case will be implicitly excluded hereafter.*

Roughly speaking, $\sigma \geq 0$ describes the fluctuations of the size, the constant $c \in \mathbb{R}$ represents the deterministic dilation (resp. erosion) coefficient when $c > 0$ (resp. $c < 0$). The measure ν is called the *dislocation measure*. Roughly speaking, for every $\mathbf{s} \in \mathcal{S}$, a fragment of size $x > 0$ splits into a sequence of fragments $x\mathbf{s}$ at rate $\nu(d\mathbf{s})$. The constant $\theta \in \mathbb{R}$ characterizes the speed at which the size of a fragment evolves towards (when $\theta > 0$) or away from (when $\theta < 0$) the value 1 (since the central location of an OU type process is 0).

Recall that $\mathbf{X}^{\downarrow}$ is associated with an OU type branching Markov process $\mathbf{Z}$, and that $\mathbf{Z}$ is the limit of a family of OU type branching Markov chains $(\mathbf{Z}^{(\ell)}, \ell \geq 0)$. We thus define for every $\ell \geq 0$ a *truncated* OU type growth-fragmentation $\mathbf{X}^{(\ell)\downarrow}$ by the exponential of $\mathbf{Z}^{(\ell)}$ (rearranged in decreasing order). In particular when $\ell = 0$, in the truncated system $\mathbf{X}^{(0)\downarrow}$ there is always at most one fragment, called *the selected fragment* of $\mathbf{X}^{\downarrow}$ (in the same sense as the selected fragment of a homogeneous fragmentation, see Section 4.2.2). Denote by X_* the size process of this selected fragment.

Lemma 4.2.12. *The process* $(\log X_*(t),\ t \geq 0)$ *is an OU type process with characteristics* (Φ_*, θ)*, where*

$$\Phi_*(q) = \frac{1}{2}\sigma^2 q^2 + cq + \int_{\mathcal{S}} (s_1^q - 1 + q(1-s_1))\, \nu(d\mathbf{s}), \quad q \geq 0.$$

In particular, the Lévy measure Λ_* *is given by*

$$\Lambda_*(dz) = \nu(\log s_1 \in dz), \quad z \in (-\infty, 0).$$

Proof. The law of $\log X_*$ can be derived directly from Lemma 4.2.6. $\square$

With the help of Theorem 4.2.9, we shall establish some fundamental properties of $\mathbf{X}^{\downarrow}$ in the rest of this section. We first prove that $\mathbf{X}^{\downarrow}$ is a time-homogeneous Markov process. In this direction, let us define a family of probability measures. Specifically, let $\alpha \in \mathrm{dom}(\kappa)$ and $\mathbf{x}^{\downarrow} = (x_1, x_2, \dots) \in \ell^{\alpha\downarrow} \subset c_o^{\downarrow}$, where $\ell^{\alpha\downarrow}$ denotes the space of decreasing null sequences with finite ℓ^{α}-norm, so $\|\mathbf{x}^{\downarrow}\|_{\ell^\alpha} := (\sum_{i=1}^{\infty} |x_i|^{\alpha})^{\frac{1}{\alpha}} < \infty$. Let $(\mathbf{X}^{[j]\downarrow}, j \in \mathbb{N})$ be a sequence of i.i.d. copies of $\mathbf{X}^{\downarrow}$. We have for every $t \geq 0$ and $q \geq \alpha(\mathrm{e}^{\theta t} \vee 1)$ that

$$\mathbb{E}\left[\sum_{j\geq 1}\sum_{i\geq 1}\left|x_j^{\mathrm{e}^{-\theta t}} X_i^{[j]}(t)\right|^q\right] = \exp\left(-\int_0^t \kappa(q\mathrm{e}^{-\theta s})ds\right)\sum_{j\geq 1}|x_j|^{q\mathrm{e}^{-\theta t}} < \infty,$$

so the elements of $\{x_j^{\mathrm{e}^{-\theta t}} X_i^{[j]}(t), i \in \mathbb{N}, j \in \mathbb{N}\}$ can be ranked in decreasing order, and we therefore write $\mathbf{P}_{\mathbf{x}^{\downarrow}}$ for the law of the resulting process.

Proposition 4.2.13 (Markov property). *Let* $s \geq 0$ *and suppose that* $\mathbf{X}^{\downarrow}(s) = \mathbf{x}^{\downarrow} = (x_1, x_2, \dots) \in \ell^{\alpha\downarrow}$ *for a certain* $\alpha \in \mathrm{dom}(\kappa)$*. Then the conditional distribution of the process* $(\mathbf{X}^{\downarrow}(t+s), t \geq 0)$ *given* $(\mathbf{X}^{\downarrow}(r), 0 \leq r \leq s)$ *is* $\mathbf{P}_{\mathbf{X}^{\downarrow}(s)}$.

This statement clearly ensures that $\mathbf{X}^{\downarrow}$ fulfills the properties **(P1)** and **(P2)** in the introduction.

Proof of Proposition 4.2.13. Write $\mathbf{Z}$ for the OU type branching Markov process associated with $\mathbf{X}^{\downarrow}$. For every $\ell \geq 0$, recall the truncated process $\mathbf{Z}^{(\ell)}$, which is an OU type branching Markov chain and we may thus define a *truncated* OU type growth-fragmentation $\mathbf{X}^{(\ell)\downarrow}$. It is plain from Definition 4.2.5 that $\mathbf{X}^{(\ell)\downarrow}$ fulfills the claimed Markov property. This observation and Theorem 4.2.9 entail that the Markov property also holds for $\mathbf{X}^{\downarrow}$. See the proof of Proposition 2 in [20] for similar arguments and we omit the details. $\square$

Combining Theorem 4.2.9 and Proposition 4.2.13, we immediately obtain the following non-negative martingales, which should be compared with the famous *additive martingales* in context of fragmentations [25] or branching random walks [28].

Proposition 4.2.14. *Let $x > 0$ and $\mathbf{X}^{\downarrow}$ be an OU type growth-fragmentation of law $\mathbf{P}_x := \mathbf{P}_{(x,0,\dots)}$.*

(i) If $\theta < 0$, then for every $q \in \mathrm{dom}(\kappa)$, the process

$$x^{-q\mathrm{e}^{-\theta t}} \exp\left(-\int_0^t \kappa(q\mathrm{e}^{-\theta s})ds\right) \sum_{i=1}^{\infty} X_i(t)^q, \quad t \geq 0 \quad \textit{is a martingale.}$$

(ii) If $\theta > 0$, then for every $\alpha \in \mathrm{dom}(\kappa)$, the process

$$x^{-\alpha} \exp\left(-\int_0^t \kappa(\alpha\mathrm{e}^{\theta s})ds\right) \sum_{i=1}^{\infty} X_i(t)^{\alpha\mathrm{e}^{\theta t}}, \quad t \geq 0 \quad \textit{is a martingale.}$$

Proposition 4.2.15 (Feller-type property). *Let $\alpha \in \mathrm{dom}(\kappa)$ and suppose that a sequence $\mathbf{x}_n^{\downarrow} \to \mathbf{x}_{\infty}^{\downarrow}$ in $\ell^{\alpha\downarrow}$, then for every $t \geq 0$ there is the weak convergence*

$$(\mathbf{P}_{\mathbf{x}_n^{\downarrow}}(s), s \in [0,t]) \Longrightarrow (\mathbf{P}_{\mathbf{x}_{\infty}^{\downarrow}}(s), s \in [0,t])$$

in the sense of finite dimensional distributions on $\ell^{q\downarrow}$ for every $q \geq \alpha(\mathrm{e}^{\theta t} \vee 1)$.

Proof of Proposition 4.2.15. Similarly as in the proof of Corollary 2 in [19], we consider a sequence $(\mathbf{X}^{[j]}, j \in \mathbb{N})$ of i.i.d. copies of $\mathbf{X}$. Fix an arbitrary $t \geq 0$ and $q \geq \alpha(\mathrm{e}^{\theta t} \vee 1)$, it follows from (4.2.13) that

$$\mathbb{E}\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left|(x_{n,j}^{\mathrm{e}^{-\theta t}} - x_{\infty,j}^{\mathrm{e}^{-\theta t}})X_i^{[j]}(t)\right|^q\right] = \exp\left(-\int_0^t \kappa(q\mathrm{e}^{-\theta s})ds\right)\sum_{j=1}^{\infty}|x_{n,j}^{\mathrm{e}^{-\theta t}} - x_{\infty,j}^{\mathrm{e}^{-\theta t}}|^q. \quad (4.2.14)$$

But then different estimations are needed for our case. More precisely, if $\theta > 0$, as the function $x \mapsto x^{\mathrm{e}^{-\theta t}}$ is concave, then for every $j \geq 1$ there is

$$|x_{n,j}^{\mathrm{e}^{-\theta t}} - x_{\infty,j}^{\mathrm{e}^{-\theta t}}| \leq |x_{n,j} - x_{\infty,j}|^{\mathrm{e}^{-\theta t}}.$$

We next consider the case when $\theta < 0$. Since $\mathbf{x}_n^{\downarrow} \to \mathbf{x}_{\infty}^{\downarrow}$ in $\ell^{\alpha\downarrow}$, we may assume that for every $n \geq 1$, there is $|x_{n,j} - x_{\infty,j}| < 1$ for every $j \geq 1$, so $\|\mathbf{x}_n^{\downarrow}\|_{\ell^\infty} \leq \|\mathbf{x}_{\infty}^{\downarrow}\|_{\ell^\infty} + 1$. Therefore, with a constant $C(t) := \mathrm{e}^{-\theta t}(\|\mathbf{x}_{\infty}^{\downarrow}\|_{\ell^\infty} + 1)^{\mathrm{e}^{-\theta t}-1}$, we have for every $j \geq 1$ that

$$|x_{n,j}^{\mathrm{e}^{-\theta t}} - x_{\infty,j}^{\mathrm{e}^{-\theta t}}| \leq C(t)|x_{n,j} - x_{\infty,j}|.$$

Combining these observations and that $\mathbf{x}_n^{\downarrow} \to \mathbf{x}_{\infty}^{\downarrow}$ in $\ell^{\alpha\downarrow}$, we deduce from (4.2.14) that

$$\lim_{n\to\infty} \mathbb{E}\left[\sum_{j\geq 1}\sum_{i\geq 1}\left|(x_{n,j}^{\mathrm{e}^{-\theta t}} - x_{\infty,j}^{\mathrm{e}^{-\theta t}})X_i^{[j]}(t)\right|^q\right] = 0.$$

Write $\mathbf{x}^{\downarrow}$ and $\mathbf{y}^{\downarrow}$ for the decreasing rearrangements of sequences $\mathbf{x}$ and $\mathbf{y}$, recall the well-known fact that $\|\mathbf{x}^{\downarrow} - \mathbf{y}^{\downarrow}\|_{\ell^q}^q \leq \|\mathbf{x} - \mathbf{y}\|_{\ell^q}^q$, see e.g. Theorem 3.5 in [62]. As a consequence, there is

$$\|(x_{n,j}^{\mathrm{e}^{-\theta t}} X_i^{[j]}(t))^{\downarrow} - (x_{\infty,j}^{\mathrm{e}^{-\theta t}} X_i^{[j]}(t))^{\downarrow}\|_{\ell^q}^q \leq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left|(x_{n,j}^{\mathrm{e}^{-\theta t}} - x_{\infty,j}^{\mathrm{e}^{-\theta t}})X_i^{[j]}(t)\right|^q,$$

which entails that

$$\lim_{n\to\infty} \mathbb{E}\left[\left\|(x_{n,j}^{\mathrm{e}^{-\theta t}} X_i^{[j]}(t))^{\downarrow} - (x_{\infty,j}^{\mathrm{e}^{-\theta t}} X_i^{[j]}(t))^{\downarrow}\right\|_{\ell^q}\right] = 0.$$

From the description of $\mathbf{P}_{\mathbf{x}_n^{\downarrow}}$ and $\mathbf{P}_{\mathbf{x}_{\infty}^{\downarrow}}$, we deduce the Feller-type property. $\square$

We finally establish the regularity of the path of $\mathbf{X}^{\downarrow}$.

Proposition 4.2.16. *For every $t \geq 0$ and $q \geq 2(\mathrm{e}^{\theta t} \vee 1)$, the process $\mathbf{X}^{\downarrow}[0,t]$ possesses a càdlàg version in $\ell^{q\downarrow}$. In particular, the process $\mathbf{X}^{\downarrow}$ possesses a càdlàg version in $c_0^{\downarrow}$.*

Proof. One can follow the same arguments as in the proof of Proposition 2 in [19], where details can be found. To avoid duplication, let us only sketch the main steps here. For every $\ell \geq 0$, let $\mathbf{Z}^{(\ell)}$ be the truncated OU type branching Markov chain and $\mathbf{X}^{(\ell)}$ be its associated growth-fragmentation, then it follows plainly from the construction that $\mathbf{X}^{(\ell)}[0,t]$ possesses a càdlàg version in $\ell^{q\downarrow}$. Further, the same arguments as in Lemma 4 in [19] show that

$$\lim_{\ell\to\infty} \sup_{0\leq s\leq t} \|\mathbf{X}(s) - \mathbf{X}^{(\ell)}(s)\|_{\ell^{q\downarrow}}^q = 0 \quad \text{in probability.} \tag{4.2.15}$$

Then it follows that $\mathbf{X}[0,t]$ possesses a càdlàg version in $\ell^{q\downarrow}$. $\square$

As a consequence of the Feller-type property and the càdlàg path, we deduce that $\mathbf{X}^{\downarrow}$ fulfills the strong Markov property by a standard argument (approximate a general stopping time by a decreasing sequence of simple stopping times, and the Markov property holds for simple stopping times).

4.2.6 Proofs of Lemma 4.2.7 and Theorem 4.2.9

In this section, we complete the proofs of Lemma 4.2.7 and Theorem 4.2.9. The key idea is to use the following decomposition of OU type branching Markov chains, which is motived from Lemma 2 in [19].

A decomposition of OU type branching Markov chains Let $\mathbf{Z}$ be an OU type branching Markov chain $\mathbf{Z}$ with characteristics (σ, c, μ, θ), so $\mu(\mathcal{R} \setminus \mathcal{R}_1) < \infty$ and (4.2.8) holds. Then $\mathbf{Z}$ is closely related to a particle system $\mathbf{W}$, which we shall call **a branching random walk with an attractor**, whose law is determined by θ and μ in the following way. The system $\mathbf{W}$ is similar to a branching random walk, but with an attractor at position 0, which attracts (resp. repels) the particles if $\theta > 0$ (resp. $\theta < 0$). More precisely, an initial particle is located at $z \in \mathbb{R}$ at time 0, whose position at time $t \geq 0$ is $\mathrm{e}^{-\theta t} z$. This particle dies after an exponential time $\tau_\emptyset$ with parameter $\mu(\mathcal{R} \setminus \mathcal{R}_1)$ and splits into a sequence of particles scattered on $\mathbb{R}$, whose positions relative to its death point are distributed according to the conditional probability $\mu(\cdot \mid \mathcal{R} \setminus \mathcal{R}_1)$. Each child moves and reproduces independently of one another as the ancestor: that is, if a child is born at time $s \geq 0$ with initial position $y \in \mathbb{R}$, then its position at time $t \geq s$ is $\mathrm{e}^{-\theta(t-s)} y$, and after an exponential time with parameter $\mu(\mathcal{R} \setminus \mathcal{R}_1)$, it branches into a cloud of children relative to its death point according to $\mu(\cdot \mid \mathcal{R} \setminus \mathcal{R}_1)$. Denote the positions of the particles alive at time $t \geq 0$ by $\mathbf{W}(t)$ and write $P_z^{\mathbf{W}}$ for the law of the process $(\mathbf{W}(t), t \geq 0)$ with $\mathbf{W}(0) = \{\!\{z\}\!\}$. It is plain from the construction that under $P_0^{\mathbf{W}}$, the process $(\mathrm{e}^{-\theta t} z + \mathbf{W}(t))_{t \geq 0}$ has the law of $P_z^{\mathbf{W}}$.

We also notice that the process

$$\tilde{\mathbf{W}}(t) := \mathrm{e}^{\theta t} \mathbf{W}(t), \qquad t \geq 0$$

is a (continuous time) branching random walk in a time-inhomogeneous environment in the following sense. Specifically, a particle branches at rate $\mu(\mathcal{R} \setminus \mathcal{R}_1)$, and if a branching happens at (global) time $t \geq 0$, then the relative locations of its children are distributed according to $\mathrm{e}^{\theta t} \mathbf{r}$, where $\mathbf{r}$ has distribution $\mu(\cdot \mid \mathcal{R} \setminus \mathcal{R}_1)$.

An OU type branching Markov chain $\mathbf{Z}$ can be viewed as a system $\mathbf{W}$ (under $P_0^{\mathbf{W}}$) superposing i.i.d. OU type processes in the following sense.

Lemma 4.2.17. *In the notation above, suppose that $\mu(\mathcal{R} \setminus \mathcal{R}_1) < \infty$. Fix a time $t \geq 0$ and write $\mathbf{W}(t) = \{\!\{W_i : \ i \in I\}\!\}$. Then there exists a family of real valued random variables $(\beta_i)_{i \in I}$ such that the multiset*

$$\{\!\{W_i + \beta_i, \ i \in I\}\!\}$$

has the same law as $\mathbf{Z}(t)$, and conditionally on $\mathbf{W}(t)$, each β_i has Laplace transform

$$\mathbb{E}\left[\exp\left(q\beta_i\right)\right] = \exp\left(\int_0^t \psi(q\mathrm{e}^{-\theta s}) ds\right), \quad q \geq 0$$

where ψ is given by (4.2.9).

Proof. We use a similar argument as the proof of Lemma 2 in [19].

Let $\partial\mathcal{U}$ be the set of infinite sequences of positive integers. For every $\bar{u} = (u_1, u_2, \ldots) \in \partial\mathcal{U}$ and $i \geq 0$, write $\bar{u}_i := (u_1, u_2, \ldots u_i) \in \mathbb{N}^i$ ($\bar{u}_0 := \emptyset$ by convention). Recall that $\mathbf{Z}$ is defined by

Definition 4.2.5. In that framework, define recursively a sequence $(\tilde{a}_{\bar{u}_j})_{j\geq 0}$ such that $\tilde{a}_{\bar{u}_0} := 0$ and

$$\tilde{a}_{\bar{u}_{j+1}} := \mathrm{e}^{-\theta\lambda_{\bar{u}_j}}\tilde{a}_{\bar{u}_j} + Z_{\bar{u}_j}(\lambda_{\bar{u}_j}).$$

We further define a process $Z_{\bar{u}}$ by

$$Z_{\bar{u}}(t) := \mathrm{e}^{-\theta(t-b_{\bar{u}_j})}\tilde{a}_{\bar{u}_j} + Z_{\bar{u}_j}(t-b_{\bar{u}_j}) \quad \text{for } t \in [b_{\bar{u}_j}, b_{\bar{u}_j}+\lambda_{\bar{u}_j}) \text{ with } j \geq 0. \tag{4.2.16}$$

Then it follows from the simple Markov property of OU type processes that each $Z_{\bar{u}}$ is an OU type process with characteristics (ψ, θ). For every $\bar{u} \in \partial\mathcal{U}$, let $\eta_{\bar{u}}$ be a (time-inhomogeneous) compound Poisson process which makes a jump of size $\mathrm{e}^{\theta b_{\bar{u}_i}}\Delta a_{\bar{u}_i}$ at time $b_{\bar{u}_i}$ for every $i \geq 0$, i.e.

$$\eta_{\bar{u}}(t) := \sum_{i=0}^{j} \exp\left(\theta b_{\bar{u}_i}\right)\Delta a_{\bar{u}_i} \quad \text{for } t \in [b_{\bar{u}_j}, b_{\bar{u}_j}+\lambda_{\bar{u}_j}) \text{ with } j \geq 0.$$

We next equip the edges of $\mathcal{U}$ with lengths, such that for every $u \in \mathcal{U}$ and $j \in \mathbb{N}$, the length of the edge connecting u and uj is λ_u, so the distance between each $u \in \mathcal{U}$ and the root $\emptyset$ in b_u. Cutting the tree $\mathcal{U}$ at height $t > 0$ (distance from the root) yields $L \subset \mathcal{U}$, i.e. $u \in L$ iff $b_u \leq t < b_u + \lambda_u$. Each $v \in L$ naturally corresponds to a subset $B_v \subset \partial\mathcal{U}$, that consists of all those $\bar{u} \in \partial\mathcal{U}$ stemming from v, and it is clear that the values $\eta_{\bar{u}}(t)$ (resp. $Z_{\bar{u}}(t)$) are the same of all $\bar{u} \in B_v$. So we define unambiguously

$$\eta_{B_v}(t) := \eta_{\bar{u}}(t) \quad \text{and} \quad Z_{B_v}(t) := Z_{\bar{u}}(t), \qquad \bar{u} \in B_v.$$

We also observe that the family $(B_v, v \in L)$ are disjoint, which form a partition of $\partial\mathcal{U}$. Since for every $j \geq 0$ there is the identity

$$a_{\bar{u}_j} = \tilde{a}_{\bar{u}_j} + \sum_{i=0}^{j} \exp\left(-\theta(b_{\bar{u}_j} - b_{\bar{u}_i})\right)\Delta a_{\bar{u}_i} = \tilde{a}_{\bar{u}_j} + \exp\left(-\theta b_{\bar{u}_j}\right)\sum_{i=0}^{j}\exp\left(\theta b_{\bar{u}_i}\right)\Delta a_{\bar{u}_i},$$

then for every $t \geq 0$ we have the identity that

$$\mathbf{Z}(t) := \{\!\{\mathrm{e}^{-\theta(t-b_u)}a_u + Z_u(t-b_u) : \ u \in L\}\!\} = \{\!\{\mathrm{e}^{-\theta t}\eta_{B_v}(t) + Z_{B_v}(t) : \ v \in L\}\!\}.$$

Observing that $\{\!\{\mathrm{e}^{-\theta t}\eta_{B_v}(t) : \ v \in L\}\!\}$ has the same law as $\mathbf{W}(t)$, we hence deduce the claim. $\square$

We now prove Lemma 4.2.7.

Proof of Lemma 4.2.7. We shall prove the claim by checking that $\mathbf{Z}^{(\ell)}$ fulfills Definition 4.2.5 with a different genealogy. The proof is an adaptation of the arguments of Lemma 3 in [19].

Consider $\bar{1} = (1, 1, 1, \dots) \in \partial\mathcal{U}$ and denote for every $i \in \mathbb{N}$ the ancestor of $\bar{1}$ in the i-th generation by $\bar{1}_i \in \mathbb{N}^i$, with $\bar{1}_0 = \emptyset$ by convention. With the notation in Definition 4.2.5 and the proof of Lemma 4.2.17, we write $\mathbf{r}_i := \Delta a_{\bar{1}_i}$ for every $i \in \mathbb{N}$ and define $\mathbf{r}_i^{(\ell)}$ as in (4.2.10). As $\mathbf{r}_i$ has the law of $\mu(\cdot|\mathcal{R} \setminus \mathcal{R}_1)$, we easily deduce that $\mathbb{P}\left(\mathbf{r}_i^{(\ell)} \notin \mathcal{R}_1\right) = \frac{\mu^{(\ell)}(\mathcal{R}\setminus\mathcal{R}_1)}{\mu(\mathcal{R}\setminus\mathcal{R}_1)}$. Let $\bar{1}_N$ be the first node along the branch $\bar{1}$ such that $\mathbf{r}_N^{(\ell)} \notin \mathcal{R}_1$, then for all $i \leq N-1$ there is $\mathbf{r}_i^{(\ell)} \in \mathcal{R}_1$, which means that only the closest child of $\bar{1}_i$ is still alive in the system $\mathbf{Z}^{(\ell)}$ but the other children are all killed. Therefore, in the truncated system $\mathbf{Z}^{(\ell)}$ there is only one particle alive at each time before $a_{\bar{1}_N} + \lambda_{\bar{1}_N}$. We hence view the displacement of the only particle as the movement of the ancestor marked by $\emptyset$ in the truncated system $\mathbf{Z}^{(\ell)}$, until its lifetime $\lambda_\emptyset^{(\ell)} := a_{\bar{1}_N} + \lambda_{\bar{1}_N}$, and then it splits into more than one particles, located relatively to the position of $\emptyset$ at death by $\Delta a_\emptyset^{(\ell)} := \mathbf{r}_N^{(\ell)}$, which is a random variable of law $\mu^{(\ell)}(\cdot\ |\mathcal{R} \setminus \mathcal{R}_1)$. Since N has the geometric distribution with parameter $\frac{\mu^{(\ell)}(\mathcal{R}\setminus\mathcal{R}_1)}{\mu(\mathcal{R}\setminus\mathcal{R}_1)}$, from basic property of exponential random variable, we know that $\lambda_\emptyset^{(\ell)}$ has the exponential distribution with parameter $\mu(\mathcal{R} \setminus \mathcal{R}_1) \times \mathbb{P}\left(\mathbf{r}_i^{(\ell)} \notin \mathcal{R}_1\right) = \mu^{(\ell)}(\mathcal{R} \setminus \mathcal{R}_1)$.

We next investigate the distribution of the movement $Z_\emptyset^{(\ell)}$ of the ancestor $\emptyset$. By a similar discussion as in the proof of Lemma 4.2.17, the process $Z_\emptyset^{(\ell)}$ is the superposition of two OU type processes: one is $Z_{\bar{1}}$ as in (4.2.16), which is an OU type process with characteristics (ψ, θ), and the other is an OU type process driven by (N, θ), where N is a compound Poisson process on $(-\infty, 0)$ with Lévy measure

$$\mu(r_1 \in dz :\ \mathbf{r}^{(\ell)} \in \mathcal{R}_1, \mathbf{r} \notin \mathcal{R}_1), \quad z \in (-\infty, 0).$$

Therefore, we have by Lemma 4.2.1 that $Z_\emptyset^{(\ell)}$ is an OU type process with characteristics $(\psi^{(\ell)}, \theta)$ where

$$\psi^{(\ell)}(q) = \psi(q) + \int_{(-\infty,0)} (\mathrm{e}^{qz} - 1)\mu(r_1 \in dz :\ \mathbf{r}^{(\ell)} \in \mathcal{R}_1, \mathbf{r} \notin \mathcal{R}_1), \quad q \geq 0.$$

Using the fact that

$$\int_{\mathcal{R}} (1 - \mathrm{e}^{r_1})\mu^{(\ell)}(d\mathbf{r}) = \int_{\mathcal{R}} (1 - \mathrm{e}^{r_1})\mu(d\mathbf{r})$$

and that $\mathbf{r} \in \mathcal{R}_1$ implies $\mathbf{r}^{(\ell)} \in \mathcal{R}_1$, we deduce that

$$\psi^{(\ell)}(q) = \frac{1}{2}\sigma^2 q^2 + \left(c + \int_{\mathcal{R}\setminus\mathcal{R}_1} (1 - \mathrm{e}^{r_1})\mu^{(\ell)}(d\mathbf{r})\right) q + \int_{\mathcal{R}_1} \left(\mathrm{e}^{qr_1} - 1 + q(1 - \mathrm{e}^{r_1})\right)\mu^{(\ell)}(d\mathbf{r}).$$

By iterating this argument and comparing with Definition 4.2.5, we complete that proof. $\square$

We finally turn to prove Theorem 4.2.9. To this vein, we need the following lemma. For every multiset $\pi := \{\!\{\pi_i, i \in I\}\!\}$ and $q \geq 0$, we adopt the notation

$$\langle \pi, \mathrm{e}^{qz} \rangle := \sum_{i \in I} \mathrm{e}^{q\pi_i}.$$

Lemma 4.2.18. *Let* $\mathbf{W}$ *be a branching random walk with an attractor starting from* $\mathbf{W}(0) := \{\!\{0\}\!\}$. *For every* $t \geq 0$ *and* $q \geq 0$, *we have*

$$\mathbb{E}\left[\langle \mathbf{W}(t), \mathrm{e}^{qz} \rangle\right] = \exp\left(\int_0^t h(\mathrm{e}^{-\theta s} q) ds\right),$$

where

$$h(q) := \int_{\mathcal{R}\setminus\mathcal{R}_1} \Big(\sum_{i=1}^{\infty} \mathrm{e}^{qr_i} - 1\Big)\mu(d\mathbf{r}).$$

Proof. Write $\tau_\emptyset$ for the branching time of the ancestor of system $\mathbf{W}$ and $(\Delta a_i, i \in \mathbb{N})$ for the sequence of positions of the first generation at birth. Decompose at $\tau_\emptyset$ and use the branching property, then there is

$$\begin{aligned}
m(q,t) &:= \mathbb{E}\left[\langle \mathbf{W}(t), \mathrm{e}^{qz} \rangle\right] \\
&= \mathbb{E}\left[\langle \mathbf{W}(t), \mathrm{e}^{qz} \rangle \mathbf{1}_{\{\tau_\emptyset > t\}}\right] + \mathbb{E}\left[\langle \mathbf{W}(t), \mathrm{e}^{qz} \rangle \mathbf{1}_{\{\tau_\emptyset \leq t\}}\right] \\
&= \mathbb{P}\left(\tau_\emptyset > t\right) + \mathbb{E}\left[\mathbf{1}_{\{\tau_\emptyset \leq t\}} \sum_{i=1}^{\infty} \exp\left(q \Delta a_i \mathrm{e}^{-\theta(t-\tau_\emptyset)}\right) \langle \mathbf{W}^i(t-\tau_\emptyset), \mathrm{e}^{qz} \rangle\right] \\
&= \mathrm{e}^{-\mu(\mathcal{R}\setminus\mathcal{R}_1)t} + \int_0^t \mu(\mathcal{R}\setminus\mathcal{R}_1)\mathrm{e}^{-\mu(\mathcal{R}\setminus\mathcal{R}_1)s} m(q, t-s) ds \int_{\mathcal{R}\setminus\mathcal{R}_1} \sum_{i=1}^{\infty} \frac{\exp\left(q\mathrm{e}^{-\theta(t-s)} r_i\right)}{\mu(\mathcal{R}\setminus\mathcal{R}_1)} \mu(d\mathbf{r}) \\
&= \mathrm{e}^{-\mu(\mathcal{R}\setminus\mathcal{R}_1)t} + \int_0^t \mathrm{e}^{-\mu(\mathcal{R}\setminus\mathcal{R}_1)s} m(q, t-s) \left(h(\mathrm{e}^{-\theta(t-s)} q) + \mu(\mathcal{R}\setminus\mathcal{R}_1)\right) ds,
\end{aligned}$$

where $(\mathbf{W}^i, i \in \mathbb{N})$ are independent copies of $\mathbf{W}$, further independent of $\tau_\emptyset$ and $(\Delta_i, i \in \mathbb{N})$. Changing variable in the integral by $t - s \mapsto s$, we have that

$$\mathrm{e}^{\mu(\mathcal{R}\setminus\mathcal{R}_1)t} m(q,t) = 1 + \int_0^t \mathrm{e}^{\mu(\mathcal{R}\setminus\mathcal{R}_1)s} m(q,s)(h(q\mathrm{e}^{-\theta s}) + \mu(\mathcal{R}\setminus\mathcal{R}_1)) ds.$$

Solving this integral equation with initial condition $m(q, 0) = 1$, we have the desired identity. $\square$

Proof of Theorem 4.2.9. We first suppose that $\mu(\mathcal{R}\setminus\mathcal{R}_1) < \infty$, then it follows from Lemma 4.2.17 and Lemma 4.2.18 that for every $q \geq 0$ there is

$$\mathbb{E}\left[\langle \mathbf{Z}(t), \mathrm{e}^{qz} \rangle\right] = \exp\left(\int_0^t \psi(\mathrm{e}^{-\theta s} q) ds\right) \exp\left(\int_0^t h(\mathrm{e}^{-\theta s} q) ds\right) = \exp\left(\int_0^t \kappa(\mathrm{e}^{-\theta s} q) ds\right),$$

where κ is defined by (4.2.12) and plainly $\kappa = \psi + h$.

Now we consider $\mu(\mathcal{R}\setminus\mathcal{R}_1)=\infty$ (but μ fulfills (4.2.8)). For every $\ell\geq 0$, recall the truncated process $\mathbf{Z}^{(\ell)}$ with characteristics $(\sigma,c,\mu^{(\ell)},\theta)$, then the cumulant of $\mathbf{Z}^{(\ell)}$ defined as in (4.2.12) is

$$\kappa^{(\ell)}(q)=\frac{1}{2}\sigma^2q^2+cq+\int_{\mathcal{R}}\left(\mathrm{e}^{qr_1}+\sum_{i=2}^{\infty}\mathbf{1}_{\{r_i>-\ell\}}\mathrm{e}^{qr_i}-1+q(1-\mathrm{e}^{r_1})\right)\mu(d\mathbf{r}),\quad q\geq 0.$$

Since μ fulfills (4.2.8), then we have $\mu^{(\ell)}(\mathcal{R}\setminus\mathcal{R}_1)<\infty$ and thus for $q\geq\alpha(\mathrm{e}^{\theta t}\vee 1)$ we have

$$\mathbb{E}\left[\langle\mathbf{Z}^{(\ell)}(t),\mathrm{e}^{qz}\rangle\right]=\exp\left(\int_0^t\kappa^{(\ell)}(\mathrm{e}^{-\theta s}q)ds\right).$$

Letting $\ell\to\infty$, it is plain that for every $p\geq\alpha$, there is

$$\lim_{\ell\to\infty}\uparrow\kappa^{(\ell)}(p)=\kappa(p)<0,$$

we hence deduce the claim by monotone convergence. $\square$

4.3 Properties of OU type growth-fragmentations

We continue to study properties of OU type growth-fragmentations. Throughout this section, let $\mathbf{X}^{\downarrow}:=(\mathbf{X}^{\downarrow}(t)=(X_1(t),X_2(t),\ldots),\ t\geq 0)$ be an OU type growth-fragmentation with characteristics (σ,c,ν,θ) and denote κ for its cumulant.Without loss of generality, we always assume that $\mathbf{X}^{\downarrow}(0)=(1,0,0,\ldots)$.

4.3.1 Growth-fragmentation equations

The evolution of the mean value of an OU type growth-fragmentation can be described by a growth-fragmentation equation.

Proposition 4.3.1. *Suppose that $\theta>0$. For every $t\geq 0$ and $f\in\mathcal{C}_c^{\infty}(0,\infty)$ (the space of C^{∞} functions on $(0,\infty)$ with compact supports), we have that $\mathbb{E}\left[\sum_{i=1}^{\infty}f(X_i(t))\right]<\infty$ and thus define a Radon measure $\rho_{\mathbf{X}^{\downarrow}}(t)$ on $(0,\infty)$ by*

$$\left\langle\rho_{\mathbf{X}^{\downarrow}}(t),f\right\rangle:=\mathbb{E}\left[\sum_{i=1}^{\infty}f(X_i(t))\right].$$

Then $\left\langle\rho_{\mathbf{X}^{\downarrow}}(0),f\right\rangle=\mathbb{E}\left[\sum_{i=1}^{\infty}f(X_i(0))\right]=f(1)$ and $(\rho_{\mathbf{X}^{\downarrow}}(t),t\geq 0)$ solves the equation

$$\left\langle\rho_{\mathbf{X}^{\downarrow}}(t),f\right\rangle=f(1)+\int_0^t\left\langle\rho_{\mathbf{X}^{\downarrow}}(r),\mathcal{L}f\right\rangle dr,\tag{4.3.1}$$

where

$$\mathcal{L}f(x) := \frac{1}{2}\sigma^2 x^2 f''(x) + \Big(c + \frac{1}{2}\sigma^2 - \theta\log x\Big) x f'(x) \\ + \int_{\mathcal{S}} \Big(\sum_{i=1}^{\infty} f(xs_i) - f(x) + xf'(x)(1-s_1)\Big)\nu(d\mathbf{s}). \qquad (4.3.2)$$

See [27] and Corollary 3.12 in [21] for similar results for self-similar growth-fragmentations. The proof of the latter relies on a remarkable spinal-decomposition. However, we shall use a different approach, by first dealing with the truncated system and then passing to the limit.

To prove Proposition 4.3.1 we start with a lemma that shows that $\mathcal{L}f$ is well-defined and continuous.

Lemma 4.3.2. *For every $f \in \mathcal{C}_c^\infty(0,\infty)$, $\mathcal{L}f$ is a continuous function on $(0,\infty)$ and is identically zero in some neighborhood of zero. Furthermore, $\mathcal{L}f(x) = o(x^q)$ as $x \to \infty$ for every $q \geq 2$.*

Proof. Set $\mathcal{L}_1 f(x) := \mathcal{L}f(x) + \theta\log(x)xf'(x)$, then we know from Lemma 2.1 in [27] [1] that $\mathcal{L}_1 f$ is continuous on $(0,\infty)$, identically zero in some neighborhood of zero and $\mathcal{L}_1 f(x) = o(x^q)$ as $x \to \infty$ for every $q \geq 2$. It follows plainly that the same properties hold for $\mathcal{L}f$. □

We next prove Proposition 4.3.1 for the finite branching case in the context of an OU type branching Markov chain.

Lemma 4.3.3. *Suppose that $\mathbf{Z}$ is an OU type branching Markov chain with characteristics (σ, c, μ, θ) such that $\mu(\mathcal{R}\setminus\mathcal{R}_1) < \infty$. For every $t \geq 0$, we may associated a Radon measure $\rho_{\mathbf{Z}}(t)$ such that for all $g \in \mathcal{C}_c^\infty(\mathbb{R})$, the space of C^∞ functions on $\mathbb{R}$ with compact supports, there is $\big\langle \rho_{\mathbf{Z}}(t), g\big\rangle := \mathbb{E}\left[\big\langle \mathbf{Z}(t), g\big\rangle\right]$. Then $(\rho_{\mathbf{Z}}(t), t \geq 0)$ is a solution of the equation*

$$\big\langle \rho_{\mathbf{Z}}(t), g\big\rangle = g(0) + \int_0^t \langle \rho_{\mathbf{Z}}(s), \mathcal{L}_{\mathbf{Z}} g\rangle ds,$$

where

$$\mathcal{L}_{\mathbf{Z}} g(z) = \frac{1}{2}\sigma^2 g''(z) + cg'(z) - \theta z g'(z) + \int_{\mathcal{R}} \Big(\sum_{i=1}^{\infty} g(z + r_i) - g(z) + (1 - \mathrm{e}^{r_1})g'(z)\Big)\mu(d\mathbf{r}).$$

Proof. Recall the decomposition of $\mathbf{Z}$ in Lemma 4.2.17 and the branching random walk description of $(\tilde{\mathbf{W}}(t) := \mathrm{e}^{\theta t}\mathbf{W}(t), t \geq 0)$. By conditioning on $\mathbf{W}(t) := \{\!\{W_i,\ i \in I\}\!\}$ we have for every $g \in \mathcal{C}_c^\infty(\mathbb{R})$ that

$$\mathbb{E}\left[\big\langle \mathbf{Z}(t), g\big\rangle\right] = \mathbb{E}\left[\sum_{i\in I} g(\mathrm{e}^{-\theta t}(\mathrm{e}^{\theta t} W_i) + \beta_i)\right] = \mathbb{E}\left[\big\langle \tilde{\mathbf{W}}(t), P_t g\big\rangle\right], \qquad (4.3.3)$$

[1] though Lemma 2.1 in [27] is only concerned with the case when ν is binary and conservative, the same arguments work under our more general setting.

where $(P_t)_{t\geq 0}$ denotes the semigroup of an OU type process with characteristics (ψ, θ) and we have used the scaling property (4.2.4) for the last equality. Since the infinitesimal generator $\mathcal{A}$ of $(P_t)_{t\geq 0}$ has domain containing $C_c^2(\mathbb{R})$ ([74]), the space of C^2 functions with compact supports, and is given by

$$\begin{aligned}\mathcal{A}g =& \frac{1}{2}\sigma^2 g''(z) + \left(c + \int_{\mathcal{R}\setminus\mathcal{R}_1}(1-\mathrm{e}^{r_1})\mu(d\mathbf{r})\right) g'(z) - \theta z g'(z) \\ &+ \int_{\mathcal{R}_1}\left(g(z+r_1) - g(z) + (1-\mathrm{e}^{r_1})g'(z)\right)\mu(d\mathbf{r}),\end{aligned}$$

we know that $\frac{\partial}{\partial t}P_t g = P_t\mathcal{A}g = \mathcal{A}P_t g$ for every $g \in \mathcal{C}_c^\infty(\mathbb{R})$. Using the classic stochastic analysis and the Poissonian construction of the branching random walk $\tilde{\mathbf{W}}$, we deduce for every $g \in \mathcal{C}_c^\infty(\mathbb{R})$ that

$$\begin{aligned}&\mathbb{E}\left[\langle \mathbf{Z}(t), g\rangle\right] - g(0) \\ &= \mathbb{E}\left[\langle \tilde{\mathbf{W}}(t), P_t g\rangle\right] - g(0) \\ &= \mathbb{E}\left[\int_0^t \frac{\partial}{\partial t}\langle \tilde{\mathbf{W}}(s), P_s g\rangle ds\right] + \mathbb{E}\left[\sum_{0\leq s\leq t}\left(\langle \tilde{\mathbf{W}}(s), P_s g\rangle - \langle \tilde{\mathbf{W}}(s-), P_s g\rangle\right)\right] \\ &= \mathbb{E}\left[\int_0^t \langle \tilde{\mathbf{W}}(s), P_s\mathcal{A}g\rangle ds\right] + \int_0^t ds \int_{\mathcal{R}\setminus\mathcal{R}_1}\sum_{k=1}^\infty \mathbb{E}\left[\sum_{w\in\tilde{\mathbf{W}}(s-)}\left(P_s g(w+\mathrm{e}^{\theta s}r_k) - P_s g(w)\right)\right]\mu(d\mathbf{r}) \\ &= \int_0^t \mathbb{E}\left[\langle \mathbf{Z}(s), \mathcal{A}g\rangle\right] ds + \int_0^t ds\int_{\mathcal{R}\setminus\mathcal{R}_1}\sum_{k=1}^\infty \mathbb{E}\left[\sum_{z\in\mathbf{Z}(s-)}(g(z+r_k) - g(z))\right]\mu(d\mathbf{r}) \\ &= \int_0^t \mathbb{E}\left[\langle \mathbf{Z}(s), \mathcal{L}_{\mathbf{Z}}g\rangle\right] ds,\end{aligned}$$

where we have used again (4.3.3) for the fourth equality. This entails the claim. □

We are now ready to prove Proposition 4.3.1.

Proof of Proposition 4.3.1. It suffices to prove for the case when $f \in \mathcal{C}_c^\infty(0,\infty)$ is further non-negative. For every $t \geq 0$ and $q \geq 2(\mathrm{e}^{\theta t}\vee 1)$, we deduce from Lemma 4.3.2 and Theorem 4.2.9 that there exists a constant $C > 0$ such that for all $r \in [0, t]$,

$$\mathbb{E}\left[\sum_{i=1}^\infty |\mathcal{L}f(X_i(r))|\right] \leq C\mathbb{E}\left[\sum_{i=1}^\infty X_i(r)^q\right] = C\exp\left(\int_0^r \kappa(q\mathrm{e}^{-\theta s})ds\right),$$

which entails that

$$\int_0^t \mathbb{E}\left[\sum_{i=1}^{\infty} |\mathcal{L}f(X_i(r))|\right] dr < \infty. \tag{4.3.4}$$

By Theorem 4.2.9 we also have $\mathbb{E}_x\left[\sum_{i=1}^{\infty} f(X_i(t))\right] < \infty$.

Let us next consider for every $\ell > 0$ the truncated OU type growth-fragmentation $\mathbf{X}^{(\ell)}$ and the OU type branching Markov chain $\log \mathbf{X}^{(\ell)}$. Apply Lemma 4.3.3 to $\log \mathbf{X}^{(\ell)}$, it is straightforward to deduce that for every $t \geq 0$ there is

$$\mathbb{E}\left[\langle \mathbf{X}^{(\ell)}(t), f\rangle\right] = f(1) + \mathbb{E}\left[\int_0^t \langle \mathbf{X}^{(\ell)}(s), \mathcal{L}^{(\ell)} f\rangle ds\right],$$

where

$$\begin{aligned}\mathcal{L}^{(\ell)} f(y) =& \frac{1}{2}\sigma^2 y^2 f''(y) + \left(c + \frac{1}{2}\sigma^2 - \theta \log y\right) y f'(y) \\ &+ \int_{\mathcal{S}} \left(f(ys_1) - f(y) + \sum_{i\geq 2} f(ys_i)\mathbf{1}_{\{s_i \geq \mathrm{e}^{-\ell}\}} + yf'(y)(1-s_1)\right)\nu(d\mathbf{s}).\end{aligned}$$

Letting $\ell \to \infty$, we immediately check by monotone convergence that (f is non-negative)

$$\lim_{\ell\to\infty} \uparrow \mathcal{L}^{(\ell)} f(y) = \mathcal{L}f(y), \quad y \geq 0.$$

Recall from (4.2.15) that $\lim_{l\to\infty} \mathbf{X}^{(\ell)}(t) = \mathbf{X}(t)$, we hence obtain by dominated convergence (ensured by (4.3.4)) that

$$\lim_{\ell\to\infty} \int_0^t \mathbb{E}\left[\langle \mathbf{X}^{(\ell)}(r), \mathcal{L}^{(\ell)} f\rangle\right] dr = \int_0^t \mathbb{E}\left[\langle \mathbf{X}(r), \mathcal{L}f\rangle\right] dr.$$

On the other hand, we deduce by monotone convergence that

$$\lim_{\ell\to\infty} \uparrow \mathbb{E}\left[\langle \mathbf{X}^{(\ell)}(t), f\rangle\right] = \mathbb{E}\left[\langle \mathbf{X}(t), f\rangle\right].$$

So we conclude that

$$\mathbb{E}\left[\langle \mathbf{X}(t), f\rangle\right] = f(1) + \int_0^t \mathbb{E}\left[\langle \mathbf{X}(r), \mathcal{L}f\rangle\right] dr,$$

which means that $\rho_{\mathbf{X}}$ is indeed a Radon measure on $(0, \infty)$ and is a solution of (4.3.1). □

4.3.2 A law of large numbers for the inward case

In this subsection we fix an OU type growth-fragmentation $\mathbf{X}^{\downarrow}$ with characteristics (σ, c, ν, θ) and cumulant κ, and always suppose that $\mathbf{X}^{\downarrow}$ is inward, i.e. $\theta > 0$. We shall study the asymptotic behaviors and obtain a law of large numbers (Corollary 4.3.7). Roughly speaking, the average of the sizes of the fragments converges to a stationary distribution as time tends to infinity.

Before stating our result, let us discuss the required assumptions. To make sense of this law of large numbers, we naturally need that the number of fragments is finite and non-zero at any finite time. In this direction, we suppose in this subsection that

$$\kappa(0) = \int_{\mathcal{S}} (\#\mathbf{s} - 1)\nu(d\mathbf{s}) < \infty, \tag{4.3.5}$$

where $\#\mathbf{s} := \sum_{i=1}^{\infty} \mathbf{1}_{\{s_i>0\}}$ and κ is the cumulant defined as in (4.4.1). Denote

$$\mathcal{S}_1 := \{\mathbf{s} \in \mathcal{S} : \ s_1 > 0, s_2 = s_3 = \ldots = 0\},$$

then (4.4.1) forces that $\nu(\mathcal{S} \setminus \mathcal{S}_1) < \infty$. So the branching rate is finite and on average a finite number of child particles are generated in each splitting event. For $t \geq 0$, we denote the number of particles at time t by

$$N(t) := \sum_{i=1}^{\infty} \mathbf{1}_{\{X_i(t) \neq 0\}}.$$

Under condition (4.3.5), the process $(N(t), t \geq 0)$ is simply a branching process, see e.g. [6] for basic properties of this process. We hence suppose that

$$\frac{1}{\nu(\mathcal{S} \setminus \mathcal{S}_1)} \int_{\mathcal{S} \setminus \mathcal{S}_1} \#\mathbf{s} \ \nu(d\mathbf{s}) > 1, \tag{4.3.6}$$

which is known as the supercritical condition, then the extinction probability

$$\mathbb{P}\left(\lim_{t\to\infty} N(t) = 0\right) < 1.$$

Note that if (4.3.6) does not hold, then the system goes almost surely extinct. As a special case of Proposition 4.2.14, the process

$$M_t := \mathrm{e}^{-\kappa(0)t} N(t), \quad t \geq 0$$

is a non-negative martingale that converges almost surely to M_∞ as $t \to \infty$. Let us recall a well-known martingale convergence result.

Lemma 4.3.4 (Theorem 5 in [29]). *Suppose that there exists $\gamma \in (1, 2]$ such that*

$$\int_{\mathcal{S} \setminus \mathcal{S}_1} (\#\mathbf{s})^\gamma \nu(d\mathbf{s}) < \infty. \tag{4.3.7}$$

Then the martingale M_t converges to M_∞ almost surely and in $L^\gamma(\mathbb{P})$. Further, conditionally on non-extinction, the limit M_∞ is strictly positive.

We hence suppose that (4.3.7) holds. In particular, we know from Lemma 4.3.4 that M_∞ is bounded in $L^\gamma(\mathbb{P})$, i.e. there exists $C_\gamma > 0$ such that

$$\sup_{t\geq 0} \mathbb{E}\left[M_t^\gamma\right] < C_\gamma. \tag{4.3.8}$$

Note that (4.3.7) is also the necessary condition for M_t to be finite in $L^\gamma(\mathbb{P})$ (Corollary III 6.1 in [6]).

The last assumption is that

$$\int_{\mathcal{S}} \sum_{i=1}^{\infty} \mathbf{1}_{\{s_i<\frac{1}{2}\}} \log(|\log s_i|) d\nu(d\mathbf{s}) < \infty. \tag{4.3.9}$$

To understand this condition, we state the following lemma, which extends Lemma 3.1 in [27] (for the case when ν is binary and conservative).

Lemma 4.3.5. *For every $\alpha \in \mathrm{dom}(\kappa)$, there exists a SNLP ξ_α with Laplace exponent*

$$\Phi_\alpha(q) := \kappa(q+\alpha) - \kappa(\alpha), \quad q \geq 0.$$

Specifically, the SNLP ξ_α has characteristics $(\sigma_\alpha, c_\alpha, \Lambda_\alpha, 0)$, where $\sigma_\alpha = \sigma$,

$$c_\alpha = c + \sigma^2\alpha + \int_{\mathcal{S}} \Big((1-s_1) - \sum_{i=1}^{\infty} s_i^\alpha (1-s_i)\Big)\nu(d\mathbf{s}),$$

and the Lévy measure Λ_α is a measure on $(-\infty, 0)$ such that for every bounded measurable function h on $(-\infty, 0)$ there is

$$\int_{(-\infty,0)} h(z)\Lambda_\alpha(dz) = \int_{\mathcal{S}} \sum_{i=1}^{\infty} s_i^\alpha h(\log s_i)\nu(d\mathbf{s}).$$

Proof. We first claim that Λ_α is a Lévy measure that satisfies (4.2.11). Indeed, since $\alpha \in \mathrm{dom}(\kappa)$ and ν satisfies (4.2.11), we have that

$$\int_{(-\infty,0)} (1-\mathrm{e}^z)^2\Lambda_\alpha(dz) = \int_{\mathcal{S}} \sum_{i=1}^{\infty} s_i^\alpha (1-s_i)^2 \nu(d\mathbf{s}) \leq \int_{\mathcal{S}} \sum_{i=2}^{\infty} s_i^\alpha \nu(d\mathbf{s}) + \int_{\mathcal{S}} (1-s_1)^2\nu(d\mathbf{s}) < \infty.$$

We next check that c_α is finite. Notice that $(1-s_1^\alpha) \leq (\alpha \vee 1)(1-s_1)$, we hence deduce from (4.2.11) that

$$\int_{\mathcal{S}} (1-s_1)(1-s_1^\alpha)\nu(d\mathbf{s}) \leq \int_{\mathcal{S}} (\alpha\vee 1)(1-s_1)^2\nu(d\mathbf{s}) < \infty.$$

As $\alpha \in \mathrm{dom}(\kappa)$ entails that $\int_{\mathcal{S}} \sum_{i=2}^{\infty} s_i^\alpha \nu(d\mathbf{s}) < \infty$, we conclude that $c_\alpha < \infty$. Therefore, there exists a SNLP ξ_α with characteristics $(\sigma_\alpha, c_\alpha, \Lambda_\alpha, 0)$. It is straightforward to check that ξ_α indeed has Laplace exponent Φ_α, which completes the proof. $\square$

In particular, as $\kappa(0) < \infty$, then it follows from Lemma 4.3.5 that

$$\Phi_0(q) := \kappa(q) - \kappa(0), \quad q \geq 0 \tag{4.3.10}$$

is the Laplace exponent of a certain SNLP. Then we observe from Lemma 4.2.2 that (4.3.9) is the sufficient and necessary condition that an OU type process with characteristics (Φ_0, θ) possesses a unique stationary distribution, denoted by Π_0. Let $\tilde{\Pi}_0$ be the image of Π_0 by the map $y \mapsto \mathrm{e}^y$, so $\tilde{\Pi}_0$ is also a probability measure.

We now state the main result of this section.

Theorem 4.3.6. *Suppose that* (4.3.5)*,* (4.3.6)*,* (4.3.7) *and* (4.3.9) *hold. Then for every* $x > 0$ *and continuous function* f *on* $[0, \infty)$ *with compact support, we have convergence in* $L^\gamma(\mathbf{P}_x)$ *that*

$$\lim_{t \to \infty} \mathrm{e}^{-\kappa(0)t} \sum_{i=1}^{\infty} f(X_i(t)) = \langle \tilde{\Pi}_0, f \rangle M_\infty. \tag{4.3.11}$$

Corollary 4.3.7. *Suppose that* (4.3.5)*,* (4.3.6)*,* (4.3.7) *and* (4.3.9) *hold. Conditionally on non-extinction, there is*

$$\lim_{t \to \infty} N(t)^{-1} \sum_{i=1}^{\infty} f(X_i(t)) = \langle \tilde{\Pi}_0, f \rangle \quad \textit{in probability.}$$

Proof of Corollary 4.3.7. Conditionally on non-extinction, M_∞ is strictly positive, so it follows from Lemma 4.3.4 that almost surely

$$\lim_{t \to \infty} \frac{\mathrm{e}\kappa(0)t}{N(t)} = M_\infty^{-1}.$$

Combining this and Theorem 4.3.6, we deduce the claim. □

Corollary 4.3.8. *Suppose that* (4.3.5)*,* (4.3.6)*,* (4.3.7) *and* (4.3.9) *hold. Let* $(\rho_{\mathbf{X}^\downarrow}(t), t \geq 0)$ *be a solution to the growth-fragmentation equation* (4.3.1) *given by Proposition 4.3.1, then the finite measure* $\mathrm{e}^{-\kappa(0)t}\rho_{\mathbf{X}^\downarrow}(t)$ *converges weakly to the probability measure* $\tilde{\Pi}_0$*. Further,* $\tilde{\Pi}_0$ *is a solution to the stationary equation: for every* $f \in \mathcal{C}_c^\infty(0, \infty)$*,*

$$\langle \tilde{\Pi}_0, \mathcal{L}f \rangle = \kappa(0) f, \tag{4.3.12}$$

where $\mathcal{L}$ *is as in* (4.3.2)*.*

Proof of Corollary 4.3.8. Taking expectation to (4.3.11), we deduce that $\mathrm{e}^{-\kappa(0)t}\rho_{\mathbf{X}^\downarrow}(t)$ converges vaguely to $\tilde{\Pi}_0$. We also know that $\rho_{\mathbf{X}^\downarrow}(t)\big([0, \infty)\big) = \mathbb{E}\left[N(t)\right] = \mathrm{e}^{-\kappa(0)t}$, so $\mathrm{e}^{-\kappa(0)t}\rho_{\mathbf{X}^\downarrow}(t)$ is a probability measure and thus the convergence also holds weakly.

It remains to prove that $\tilde{\Pi}_0$ is a solution to (4.3.12). Since $(\rho_{\mathbf{X}^\downarrow}(t), t \geq 0)$ is a solution to (4.3.1), we easily check that

$$\frac{\partial}{\partial t}\left\langle \mathrm{e}^{-\kappa(0)t}\rho_{\mathbf{X}^\downarrow}(t), f\right\rangle = -\kappa(0)\left\langle \mathrm{e}^{-\kappa(0)t}\rho_{\mathbf{X}^\downarrow}(t), f\right\rangle + \left\langle \mathrm{e}^{-\kappa(0)t}\rho_{\mathbf{X}^\downarrow}(t), \mathcal{L}f\right\rangle.$$

Letting $t \to \infty$, we conclude the claim. $\square$

Theorem 4.3.6 and Corollary 4.3.7 should be compared with the law of large numbers in branching diffusions [43] and the convergence results of Crump-Mode-Jagers branching processes [68, 50]. Corollary 4.3.8 is about the long time asymptotic for the solutions of growth-fragmentation equations, see [66] and references therein for similar estimates.

Remark 4.3.9. *A natural question is whether the convergence also holds almost surely. We expect that methods used in the proof of Theorem 6 in [43] might be of use, that is, by first proving along lattice times, then replacing lattice times with continuous time. This might be an interesting open question.*

The core of the proof of Theorem 4.3.6 is the following many-to-one formula.

Lemma 4.3.10. *Suppose that (4.3.5) holds. Let χ be the exponential of an OU type process with characteristics (Φ_0, θ), where Φ_0 is as in (4.3.10). Then for every continuous function f on $(0, \infty)$ with compact support and $t \geq 0$, we have*

$$\mathbb{E}\left[f(\chi(t))\right] = \mathrm{e}^{-\kappa(0)t}\mathbb{E}\left[\sum_{i=1}^{\infty} f(X_i(t))\right]. \tag{4.3.13}$$

Proof. We deduce from Theorem 4.2.9 that this equality holds for power functions $x \mapsto x^q$ for all $q \geq 0$. That the same holds more generally for all continuous functions on $(0, \infty)$ with compact support follows from standard arguments, using Stone-Weierstrass theorem. $\square$

We are now ready to prove Theorem 4.3.6.

Proof of Theorem 4.3.6. By the scaling property **(P2)**, we may assume $x = 1$ and consider $\mathbf{X}^\downarrow$ under $\mathbf{P} := \mathbf{P}_1$. Equivalently, we shall prove that for every continuous function g on $\mathbb{R}$ with compact support, we have convergence in $L^\gamma(\mathbf{P})$ that

$$\lim_{t\to\infty} \mathrm{e}^{-\kappa(0)t}\sum_{i=1}^{\infty} g(\log X_i(t)) = \langle \Pi_0, g\rangle M_\infty.$$

For simplicity, let us denote for every $t \geq 0$ that

$$U_t := \mathrm{e}^{-\kappa(0)t}\sum_{i=1}^{\infty} g(\log X_i(t)),$$

and for every $i \in \mathbb{N}$ and $s \geq 0$ that

$$Y_i := \sum_{j=1}^{\infty} \mathbf{1}_{\left\{X_j^{(i)}(s)>0\right\}} g(\mathrm{e}^{-\theta s} \log X_i(t) + \log X_j^{(i)}(s)),$$

where $(\mathbf{X}^{(i)\downarrow} := (X_1^{(i)}(t), X_2^{(i)}(t), \ldots)_{t\geq 0}, i \geq 1)$ are i.i.d. copies of $\mathbf{X}^{\downarrow}$. Let $(\mathcal{F}_t)$ be the natural filtration of $\mathbf{X}^{\downarrow}$, then using the Markov property, Proposition 4.2.13, we have the identity in law that

$$U_{t+s} - \mathbb{E}\left[U_{t+s} \mid \mathcal{F}_t\right] \overset{d}{=} \mathrm{e}^{-\kappa(0)(t+s)} \sum_{i=1}^{\infty} \mathbf{1}_{\{X_i(t)>0\}}(Y_i - \mathbb{E}\left[Y_i \mid \mathcal{F}_t\right]). \tag{4.3.14}$$

Let us recall a useful result (Lemma 1 in [29]): if $\gamma \in [1,2]$ and Z_i are independent random variables with $\mathbb{E}\left[Z_i\right] = 0$, then for every $n \in \mathbb{N} \cup \{\infty\}$ there is

$$\mathbb{E}\left[\left|\sum_{i=1}^{n} Z_i\right|^{\gamma}\right] \leq 2^{\gamma} \sum_{i=1}^{n} \mathbb{E}\left[|Z_i|^{\gamma}\right].$$

Applying this to (4.3.14), we have that

$$\mathbb{E}\left[\Big|U_{t+s} - \mathbb{E}\left[U_{t+s} \mid \mathcal{F}_t\right]\Big|^{\gamma}\right] \leq 2^{\gamma} \mathrm{e}^{-\gamma\kappa(0)(t+s)} \sum_{i=1}^{\infty} \mathbb{E}\left[\mathbf{1}_{\{X_i(t)>0\}} \Big|Y_i - \mathbb{E}\left[Y_i \mid \mathcal{F}_t\right]\Big|^{\gamma}\right].$$

We next estimate the right-hand side. Using Jensen's inequality (the finite form) and then conditional Jensen's inequality, we find for every $i \in \mathbb{N}$ that

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}_{\{X_i(t)>0\}} \Big|Y_i - \mathbb{E}\left[Y_i \mid \mathcal{F}_t\right]\Big|^{\gamma}\right] \\ &\qquad \leq 2^{\gamma-1} \mathbb{E}\left[\mathbf{1}_{\{X_i(t)>0\}}(|Y_i|^{\gamma} + |\mathbb{E}\left[Y_i \mid \mathcal{F}_t\right]|^{\gamma})\right] \leq 2^{\gamma} \mathbb{E}\left[\mathbf{1}_{\{X_i(t)>0\}} |Y_i|^{\gamma}\right].\end{aligned}$$

By conditioning on $\mathcal{F}_t$ and using (4.3.8), we deduce that

$$\mathbb{E}\left[\mathbf{1}_{\{X_i(t)>0\}} |Y_i|^{\gamma}\right] \leq \|g\|_{\infty}^{\gamma} \mathbb{E}\left[\mathbf{1}_{\{X_i(t)>0\}} \Big(\sum_{j=1}^{\infty} \mathbf{1}_{\left\{X_j^{(i)}(s)\neq 0\right\}}\Big)^{\gamma}\right] \leq \|g\|_{\infty}^{\gamma} C_{\gamma} \mathrm{e}^{\gamma\kappa(0)s} \mathbb{E}\left[\mathbf{1}_{\{X_i(t)>0\}}\right].$$

Summarizing, we have for every $s, t > 0$ that

$$\mathbb{E}\left[|U_{t+s} - \mathbb{E}\left[U_{t+s} \mid \mathcal{F}_t\right]|^{\gamma}\right] \leq 2^{2\gamma} \mathrm{e}^{-\gamma\kappa(0)t} \|g\|_{\infty}^{\gamma} C_{\gamma} \mathbb{E}\left[M_t\right],$$

which converges to 0 as $t \to \infty$. Hence to prove the claim, it remains to prove that (take $s = t$)

$$\lim_{t\to\infty} \mathbb{E}\left[U_{2t} \mid \mathcal{F}_t\right] = \langle \Pi_0, g \rangle M_{\infty}, \quad \text{in } L^{\gamma}(\mathbf{P}). \tag{4.3.15}$$

To that end, we first observe that

$$\mathbb{E}\left[U_{t+s} \mid \mathcal{F}_t\right] = \mathrm{e}^{-\kappa(0)(t+s)} \sum_{i=1}^{\infty} \mathbb{E}\left[Y_i \mid \mathcal{F}_t\right].$$

Apply the many-to-one formula (4.3.13) to $\mathbb{E}\left[Y_i \mid \mathcal{F}_t\right]$, then it follows that

$$\mathrm{e}^{-\kappa(0)s}\mathbb{E}\left[Y_i \mid \mathcal{F}_t\right] = \mathbb{E}\left[g(\mathrm{e}^{-\theta s}\log x_i + \log\chi(s))\right]\Big|_{x_i = X_i(t)}, \tag{4.3.16}$$

where χ is the exponential of an OU type process with characteristics (Φ_0, θ). Consider a family of increasing compact sets $K(t) \uparrow (0,\infty)$, say $K_t := [t^{-1}, t]$. On the one hand, if we only consider those i such that $X_i(t) \notin K_t$, then it follows from (4.3.16) that

$$\mathrm{e}^{-\kappa(0)(t+s)} \sum_{i=1}^{\infty} \mathbf{1}_{\{X_i(t)\notin K_t\}}\mathbb{E}\left[Y_i \mid \mathcal{F}_t\right] \le \|g\|_\infty \mathrm{e}^{-\kappa(0)t} \sum_{i=1}^{\infty} \mathbf{1}_{\{X_i(t)\notin K_t\}},$$

where the right-hand side in bounded in $L^\gamma(\mathbf{P})$ and by the many-to-one formula has mean value

$$\|g\|_\infty \mathbb{P}\left(\chi(t) \notin K_t\right),$$

which converges to zero as $t \to \infty$. So we have by the dominated convergence that

$$\lim_{t\to\infty} \mathrm{e}^{-\kappa(0)(t+s)} \sum_{i=1}^{\infty} \mathbf{1}_{\{X_i(t)\notin K_t\}}\mathbb{E}\left[Y_i \mid \mathcal{F}_t\right] = 0 \quad \text{in } L^\gamma(\mathbf{P}). \tag{4.3.17}$$

On the other hand, if $X_i(t) \in K_t$, then it follows from (4.3.16) that

$$\left|\mathrm{e}^{-\kappa(0)s}\mathbb{E}\left[Y_i \mid \mathcal{F}_t\right] - \mathbb{E}\left[g(\log\chi(s))\right]\right| < c_g \mathrm{e}^{-\theta s}\log t, \tag{4.3.18}$$

where c_g is the Lipschitz constant of g. Recall that Π_0 is the invariant measure of $\log\chi$, so $\lim_{s\to\infty}\mathbb{E}\left[g(\log\chi(s))\right] = \langle \Pi_0, g\rangle$. We now take $s := t$ and use (4.3.18) and Lemma 4.3.4, then there is

$$\lim_{t\to\infty} \mathrm{e}^{-\kappa(0)(t+t)} \sum_{i=1}^{\infty} \mathbf{1}_{\{X_i(t)\in K_t\}}\mathbb{E}\left[Y_i \mid \mathcal{F}_t\right] = \langle \Pi_0, g\rangle M_\infty \quad \text{in } L^\gamma(\mathbf{P}). \tag{4.3.19}$$

Combining (4.3.17), (4.3.19), we then deduce (4.3.15), which completes the proof. □

4.4 Connections with Markovian growth-fragmentation processes

In this section, we first present *Markovian growth-fragmentation processes* [20] associated with exponential OU type processes, and then study their connections with OU type growth-fragmentations.

4.4.1 Markovian growth-fragmentations associated with exponential OU type processes

Let ξ be a SNLP with characteristics (σ, c, Λ, k). Recall that the Laplace exponent Φ of ξ is given by (4.2.1). We introduce $\kappa\colon [0,\infty) \to (-\infty,\infty]$ by

$$\kappa(q) := \Phi(q) + \int_{(-\infty,0)} (1-\mathrm{e}^z)^q \Lambda(dz), \quad q \geq 0. \tag{4.4.1}$$

So $\kappa \geq \Phi$. Note that κ is convex and $\kappa(q) < \infty$ for all $q \geq 2$ because of (4.2.2). The function κ is also called *cumulant*, and we shall later see that κ indeed plays a similar role as the cumulant of an OU type growth-fragmentation defined as in (4.2.12). We stress that κ does not characterize the law of ξ, see Lemma 2.1 in [76]. The cumulant κ plays a crucial role in the study of self-similar (Markovian) growth-fragmentation, see [20, 76].

Throughout this section, let Z be an OU type process with characteristics (Φ, θ) and denote

$$X(t) := \exp(Z(t)), \quad t \geq 0.$$

For every $x > 0$, write P_x for the law of X starting from $X(0) = x$.

A Markovian growth-fragmentation process associated with X can be constructed by using the approach in [76]. In that preparation, we state the following property of X. Define a function $f : [0,\infty) \times (0,\infty) \to (0,\infty)$ by

$$f(t,x) := x^{2\exp(\theta t)} R_1(t) R_2(t), \quad t \geq 0, x > 0,$$

where

$$R_1(t) := \exp\left(-\int_0^t \Phi(2\mathrm{e}^{\theta r}) dr\right)$$

and

$$R_2(t) := \exp\left(-\int_0^t \eta^{-1}\left(\kappa(2\mathrm{e}^{\theta r}) - \Phi(2\mathrm{e}^{\theta r})\right) dr\right),$$

with a constant $\eta \in (0,1)$. Note that R_2 is non-increasing.

Lemma 4.4.1. *For every $x > 0$ and $s, t \geq 0$, there are*

$$E_x \left[f(s+t, X(t)) + \sum_{0 \leq r \leq t} f(s+r, -\Delta X(r)) \right] = f(s,x) \left(\eta + (1-\eta) \frac{R_2(s+t)}{R_2(s)} \right) \leq f(s,x). \tag{4.4.2}$$

and

$$E_x \left[\sum_{0 \leq r} f(s+r, -\Delta X(r)) \right] \leq \eta f(s,x). \tag{4.4.3}$$

Proof. Applying (4.2.5) with $q = 2\exp(\theta(t+s))$, we have for every $s \geq 0$ that

$$E_x [f(s+t, X(t))] = x^{2\exp(\theta s)} \exp\left(\int_0^t \Phi(2\mathrm{e}^{\theta(t+s-r)}) dr \right) R_1(t) R_2(t) = x^{2\exp(\theta s)} R_1(s) R_2(s+t) \tag{4.4.4}$$

As (4.2.3) shows that

$$-\Delta X(r) = X(r-)(1 - \mathrm{e}^{\Delta\xi(r)}),$$

applying the compensation formula (see e.g. [12]) to the Poisson point process $\Delta\xi$, we have that

$$\begin{aligned} & E_x \left[\sum_{0 \leq r \leq t} f(s+r, -\Delta X(r)) \right] \\ &= \int_0^t \mathbb{E}_x [f(s+r, X(r))] \, dr \int_{(-\infty,0)} (1 - \mathrm{e}^z)^{2\exp(\theta(s+r))} \Lambda(dz) \\ &= \int_0^t x^{2\exp(\theta s)} R_1(s) R_2(s+r) \left(\kappa(2\mathrm{e}^{\theta(s+r)}) - \Phi(2\mathrm{e}^{\theta(s+r)}) \right) dr \\ &= \eta x^{2\exp(\theta s)} R_1(s) \left(R_2(s) - R_2(s+t) \right) \end{aligned} \tag{4.4.5}$$

where we have used (4.4.4) in the second equality. Adding (4.4.4) to (4.4.5) and using the fact that R_2 is non-increasing, we obtain (4.4.2). Letting $t \to \infty$ in (4.4.5), we also have (4.4.3). □

Lemma 4.4.1 enables us to construct a Markovian non-explosive growth-fragmentation process $\mathbf{X}$ associated with X. Specifically, we first list the jump times of X as a sequence $(t_i, i \in \mathbb{N})$ such that $(f(|\Delta X(t_i)|, t_i), i \in \mathbb{N}))$ is decreasing. In the sequel, **the i-th jump time of X** shall always refer to the i-th element t_i in this sequence.

We next construct the **cell system driven by** X, which is a family of processes indexed by the Ulam-Harris tree $\mathcal{U} := \bigcup_{i=0}^{\infty} \mathbb{N}^i$,

$$\mathcal{X} := (\mathcal{X}_u, \; u \in \mathcal{U}),$$

where each $\mathcal{X}_u$ depicts the evolution of the size of the cell indexed by u as time passes. Specifically, we fix $x > 0$, which is the initial size of the ancestor cell. Then we set the birth time of $\emptyset$

at $b_\emptyset := 0$ and let the life career $\mathcal{X}_\emptyset = (\mathcal{X}_\emptyset(t), t \geq 0)$ be a process of law P_x. Given the life path of $\mathcal{X}_\emptyset$, then we generate the first generation. For $i \in \mathbb{N}$, say the i-th jump time of $\mathcal{X}_\emptyset$ occurs at time t_i and has size $x_i := -\Delta\mathcal{X}_\emptyset(t_i)$, we then set $b_i = t_i$ and build a sequence of conditional independent processes $(\mathcal{X}_i)_{i\in\mathbb{N}}$ with respective conditional distribution P_{x_i}. We continue in this way to construct higher generations recursively. Write $\mathcal{P}_x$ for the law of this cell system $\mathcal{X}$ (recall that $x > 0$ indicates the initial size of the Eve $\emptyset$, i.e. $\mathcal{X}_\emptyset(0) = x$), the probability distribution $\mathcal{P}_x$ indeed exists and is uniquely determined by the above description.

Finally, for every $t \geq 0$ let $\mathbf{X}(t)$ be the multiset whose elements are sizes of the cells alive at time t, i.e.

$$\mathbf{X}(t) := \{\!\{\mathcal{X}_u(t - b_u) \ : \ u \in \mathcal{U}, b_u \leq t\}\!\},$$

As (4.4.2) holds, we deduce from Lemma 3.2 in [76] that $\mathbf{X}(t)$ can be ranked in decreasing order and we obtain $\mathbf{X}^{\downarrow}(t) \in \ell^{2\mathrm{e}^{\theta t}\downarrow}$. We refer to $\mathbf{X}^{\downarrow} = (\mathbf{X}^{\downarrow}(t), t \geq 0)$ as a **growth-fragmentation process driven by** X and write $\mathbf{P}_x$ for the law of $\mathbf{X}^{\downarrow}$ under $\mathcal{P}_x$.

We next investigate the family of OU type processes which give rise to the same (in law) Markovian growth-fragmentation. In order to apply results in [76] (or Chapter 3), we introduce the following notion.

Definition 4.4.2. *A pair of exponential OU type processes $(X, \tilde{X})$ is a* **bifurcator** *if it satisfies the following properties:*

(i) Let $\tau := \inf\{t \geq 0 : X(t) \neq \tilde{X}(t)\}$. There is almost surely either $\tau = \infty$ or the identity

$$X(\tau) + \tilde{X}(\tau) = X(\tau-) = \tilde{X}(\tau-).$$

(ii) (Asymmetric Markov branching property) Conditionally given $\tau > t$, the process

$$(X(r+t)X(t)^{-\exp(-\theta t)}, \tilde{X}(r+t)\tilde{X}(t)^{-\exp(-\theta t)})_{r\geq 0}$$

is a copy of $(X, \tilde{X})$; conditionally given $\tau \leq t$, the two processes $(X(r+t)X(t)^{-\exp(-\theta t)})_{r\geq 0}$ and $(\tilde{X}(r+t)\tilde{X}(t)^{-\exp(-\theta t)})_{r\geq 0}$ are independent copies of X and $\tilde{X}$ respectively.

Bifurcators were first introduced by Pitman and Winkel for *fragmentors* (exponentials of the negatives of pure-jump subordinators), see Definition 2 and 3 in [70]. This notion is then extended for general Markov processes, see Definition 3.3.7 (Definition 3.7 in [76]), and a bifurcator of exponential OU type processes indeed fulfills this definition.

Lemma 4.4.3. *Let $\tilde{X}$ be an OU type process with characteristics $(\tilde{\Phi}, \theta)$. Suppose that $\tilde{X}$ has the same cumulant κ. Then the growth-fragmentations $\mathbf{X}^{\downarrow}$ and $\tilde{\mathbf{X}}^{\downarrow}$, driven respectively by X and $\tilde{X}$, have the same finite-dimensional distributions.*

Proof. Since X and $\tilde{X}$ have the same cumulant κ, we readily know from Proposition 3.2.5 (Proposition 2.5 in [76]) that we may assume that ξ is the *switching transform* of $\tilde{\xi}$, see Lemma 3.2.2

(Lemma 2.2 in [76]) for the precise meaning. In particular, this means that for the switching time $\tau := \inf \left\{ t \geq 0 : \ \xi(t) \neq \tilde{\xi}(t) \right\}$, there is $\tau > 0$ almost surely and

$$\exp(\xi(\tau)) + \exp(\tilde{\xi}(\tau)) = \exp(\xi(\tau-)).$$

Let $\tilde{X}'$ be an independent copy of $\tilde{X}$. Denote $\tilde{x} := -\Delta X(\tau)$ and we build a process

$$\tilde{X}''(t) := \tilde{X}(t)\mathbf{1}_{\{t<\tau\}} + \tilde{x}^{\exp(-\theta(t-\tau))}\tilde{X}'(t-\tau)\mathbf{1}_{\{t\geq\tau\}}, \quad t \geq 0.$$

Using the scaling property (4.2.4) and the strong Markov property of an OU type process, one easily checks that $\tilde{X}'' \stackrel{d}{=} \tilde{X}$ and further $(X, \tilde{X}'')$ is a bifurcator in the sense of Definition 4.4.2 (so $(X, \tilde{X}'')$ fulfills Definition 3.3.7 as well). Combining this and Lemma 4.4.1, we check that the conditions of Theorem 3.3.9 (Theorem 3.7 in [76]) are fulfilled, then it follows that $\mathbf{X}^{\downarrow}$ and $\tilde{\mathbf{X}}^{\downarrow}$ have the same finite-dimensional distributions. □

4.4.2 Binary OU type growth-fragmentations

Definition 4.4.4. *A* **binary OU type growth-fragmentation process** *is an OU type growth-fragmentation whose dislocation measure ν has support on*

$$\{\mathbf{s} \in \mathcal{S} \ : \ s_1 + s_2 = 1, s_3 = s_4 = \ldots = 0\} \bigcup \{(0, 0, \ldots)\}. \tag{4.4.6}$$

In this subsection we study the relation between Markovian growth-fragmentations and the binary OU type growth-fragmentation processes. We first see that each binary OU type growth-fragmentation can be viewed as a Markovian growth-fragmentation in the following sense.

Lemma 4.4.5. *Let $\mathbf{X}^{\downarrow}$ be a binary OU type growth-fragmentation with characteristics (σ, c, ν, θ) and X_* be the selected fragment of $\mathbf{X}^{\downarrow}$. Then the Markovian growth-fragmentation associated with X_* is also a binary OU type growth-fragmentation with characteristics (σ, c, ν, θ).*

Proof. This proof is an adaptation of arguments in the proof of Proposition 3 in [20]. Specifically, let us consider for every $\ell > 0$ the truncated process $\mathbf{X}^{(\ell)\downarrow}$, which corresponds to an OU type branching Markov process with characteristics $(\sigma, c, \mu^{(\ell)}, \theta)$. Recall that $\mathbf{X}^{\downarrow}$ is binary, that is ν satisfies (4.4.6), so it follows that

$$\mu^{(\ell)}(\mathbf{r} \in \mathcal{R}_1 : r_1 < \log(1 - \mathrm{e}^{-\ell})) = \mu(\mathbf{r} \in \mathcal{R} : r_1 < \log(1 - \mathrm{e}^{-\ell}), r_2 \leq -\ell) = 0.$$

Similarly, we have $\mu^{(\ell)}(\mathbf{r} \in \mathcal{R} \setminus \mathcal{R}_1 : r_1 \geq \log(1 - \mathrm{e}^{-\ell})) = 0$. Recall from Lemma 4.2.12 that the logarithm of the selected fragment $\log X_*$ is an OU type process. We observe that $t \geq 0$ is a dislocation time if and only if $\log X_*(t) - \log X_*(t-) < \log(1 - \mathrm{e}^{-\ell})$, which is equivalent to

$\frac{|\Delta X_*(t)|}{X_*(t-)} > \mathrm{e}^{-\ell}$, and at each dislocation time t a child particle is born with initial position

$$\log X_*(t-) + \log(1 - \exp(\log X_*(t) - \log X_*(t-))) = \log(|\Delta X_*(t)|).$$

Therefore, the dynamics of $\mathbf{X}^{(\ell)}$ can be described in the following way. Let P_x be the law of the process $(x^{\exp(-\theta t)} X_*(t))_{t\geq 0}$. The ancestor of the cell system is the selected fragment X_*. At each time $t \geq 0$ when $\frac{|\Delta X_*(t)|}{X_*(t-)} > \mathrm{e}^{-\ell}$, if t is not the killing time of X_*, then a child cell is born with initial size $y := |\Delta X_*(t)|$. The size of the child particle evolves according the selected fragment of the sub-population, so has the law of P_y. We therefore obtain all the child cell processes in the first generation, they evolve independently one of the others. Iterating this argument, we find the cell processes in all generation and conclude that $\mathbf{X}^{(\ell)\downarrow}$ has the same law as a cell system associated with X_*, in which each child cell uj (together with its descendance) for $u \in \mathcal{U}$ is killed whenever its size at birth is less than or equal to $\mathrm{e}^{-\ell}$ times the size of the parent right before the birth of child. Letting $\ell \to \infty$, the claim follows from the monotonicity. $\square$

Corollary 4.4.6. *The law of a binary OU type growth-fragmentation $\mathbf{X}^{\downarrow}$ is characterized by (κ, θ).*

Proof. Suppose that another binary OU type growth-fragmentation $\tilde{\mathbf{X}}^{\downarrow}$ also has index θ and cumulant κ. Using the binary condition (4.4.6) and Lemma 4.2.12, we deduce that the respective selected fragments of $\tilde{\mathbf{X}}^{\downarrow}$ and $\mathbf{X}^{\downarrow}$ have the same law. Then it follows from Lemma 4.4.5 that $\tilde{\mathbf{X}}^{\downarrow}$ and $\mathbf{X}^{\downarrow}$ are the same (in law) OU type growth-fragmentation. Conversely, if an OU type growth-fragmentation $\tilde{\mathbf{X}}^{\downarrow}$ have the same law as $\mathbf{X}^{\downarrow}$, then it follows directly from (4.2.13) and the scaling property **(P2)** that $\tilde{\mathbf{X}}^{\downarrow}$ and $\mathbf{X}^{\downarrow}$ have the same index θ and cumulant κ. $\square$

Conversely, each Markovian growth-fragmentation driven by an exponential OU type process is a binary OU type growth-fragmentation.

Proposition 4.4.7. *Let Z be an OU type process driven by a SNLP ξ with cumulant κ. Then the Markovian growth-fragmentation $\mathbf{X}^{\downarrow}$ associated with $\exp(Z)$ is a version of the binary OU type growth-fragmentation characterized by (κ, θ). In particular, $\mathbf{X}^{\downarrow}$ possesses a càdlàg version in $c_0^{\downarrow}$ and for every $t \geq 0$ and $q \geq 2(1 \vee \mathrm{e}^{\theta t})$*

$$\mathbb{E}_x\left[\sum_{i=1}^{\infty} X_i(t)^q\right] = x^{q\mathrm{e}^{-\theta t}} \exp\left(\int_0^t \kappa(q\mathrm{e}^{-\theta s}) ds\right) < \infty.$$

Proof. We check that Z has the same cumulant as the selected fragment of a binary OU type growth-fragmentation characterized by (κ, θ). We hence deduce from Lemma 4.4.3 and Lemma 4.4.5 that $\mathbf{X}^{\downarrow}$ has the same finite-dimensional distributions as a binary OU type growth-fragmentation characterized by (κ, θ). Since an OU type growth-fragmentation has càdlàg path, we deduce that they have the same law. We complete the proof by applying Theorem 4.2.9 to $\mathbf{X}^{\downarrow}$. $\square$

Remark 4.4.8. *Write $(\sigma, c, \Lambda, k, \theta)$ for the characteristics of Z, then the Markovian growth-fragmentation $\mathbf{X}^{\downarrow}$ associated with $\exp(Z)$ is an OU type growth-fragmentation with characteristics*

$$\Big(\sigma,\ c + \int_{(-\infty, -\log 2)} (1 - 2\mathrm{e}^{z})\Lambda(dz) - k,\ \nu_2 + k\delta_{(0,0,\ldots)}\Big),$$

where ν_2 is the image of Λ by the map $z \mapsto (\max(\mathrm{e}^z, 1 - \mathrm{e}^z), \min(\mathrm{e}^z, 1 - \mathrm{e}^z), 0, \ldots)$.

We conclude this section with a corollary of Proposition 4.4.7, which is an analogue of Theorem 1.1 (for homogeneous growth-fragmentations) and Theorem 1.2 (for self-similar case) in [76].

Corollary 4.4.9. *Let $\tilde{X}$ be an OU type process with characteristics $(\tilde{\Phi}, \tilde{\theta})$, $\mathbf{X}$ and $\tilde{\mathbf{X}}$ be two Markovian growth-fragmentations driven respectively by X and $\tilde{X}$. Then the following statements are equivalent:*

- *(i) $\kappa = \tilde{\kappa}$ and $\theta = \tilde{\theta}$;*
- *(ii) X and $\tilde{X}$ can be coupled to form a bifurcator (Definition 4.4.2);*
- *(iii) the growth-fragmentations $\mathbf{X}$ and $\tilde{\mathbf{X}}$ have the same law.*

Proof. We have already obtained $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ from the proof of Lemma 4.4.3. The implication $(iii) \Rightarrow (i)$ follows from Proposition 4.4.7. □

4.5 A connection with the random recursive tree

In this section we lift from [10] a certain OU type growth-fragmentation that appears in the destruction of an infinite recursive tree. See also [67] for a related work.

An *infinite recursive tree* is a random rooted planar tree with vertices indexed by $\mathbb{N}$, constructed recursively in the following way. We start with linking the vertex 1 (the root) to the the vertex 2 by an edge denoted by e_2 and proceed by induction. For $i \geq 2$, vertex i attaches itself to a vertex chosen uniformly from $\{1, \ldots, i-1\}$, say j, by an edge e_i. We view i as a child of j, so j is the parent of i.

We destroy the infinite recursive tree by associating each e_i with an independent exponential clock and breaking each edge when its clock rings. Then the vertices of this tree split into different connected clusters. Let $\Pi(t) = (\Pi_1(t), \Pi_2(t), \ldots)$ be the resulting partition of $\mathbb{N}$ at time $t \geq 0$, such that each $\Pi_i(t)$ is the set of the vertices of a cluster at time t, and they are listed in increasing order of the smallest element of the cluster. It has been proven in [10] that for every $i \geq 1$

$$W_i(t) := \lim_{n\to\infty} n^{-\mathrm{e}^{-t}} \#\{k \leq n \ :\ k \in \Pi_i(t)\}$$

exists. Let $\mathbf{X}^R(t) := \{\!\{W_i(t), i \in \mathbb{N}\}\!\}$, which stands for the multiset of elements $W_i(t)$.

Let $\mathbf{X}^{R\downarrow}(t)$ be the sequence obtained by ordering $\mathbf{X}^{R}(t)$ in decreasing order. Results in [10] can be rewritten in our terms as follows.

Proposition 4.5.1 ([10]). *The process $\mathbf{X}^{R\downarrow}$ is a binary OU type growth-fragmentation with characteristics $(\kappa_R, 1)$ in the sense of Corollary 4.4.6, where*

$$\kappa_R(q) = q\psi(q+1) + (q-1)^{-1}, \quad q > 1,$$

with ψ denoting the digamma function, that is the logarithmic derivative of the gamma function. Equivalently, $\mathbf{X}^{R\downarrow}$ has characteristics $(0, -\gamma+1+\log 2, \nu, 1)$, where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant, and the binary dislocation measure ν is specified by

$$\nu(ds_1) = \left(s_1^{-2} + (1-s_1)^{-2}\right) ds_1, \qquad \frac{1}{2} \le s_1 < 1.$$

This is essentially a consequence of Proposition 2.3 and Theorem 3.1 in [10]. Then by Proposition 4.2.13 and Theorem 4.2.9, we recover immediately Theorem 3.4 in [10], which states the Markov property of $\mathbf{X}^{R\downarrow}$ and that for every $t \ge 0$ and $q > \mathrm{e}^t$, there is

$$\mathbb{E}\left[\sum_{i=1}^{\infty} X_i^R(t)^q\right] = \frac{q-1}{\mathrm{e}^{-t}q-1}\frac{\Gamma(q)}{\Gamma(\mathrm{e}^{-t}q)}. \tag{4.5.1}$$

Indeed, using the property of the digamma function ψ, we have

$$\exp\left(\int_0^t \mathrm{e}^{-s}q\psi(\mathrm{e}^{-s}q+1)ds\right) = \frac{\Gamma(q+1)}{\Gamma(\mathrm{e}^{-t}q+1)}.$$

An easy calculation shows that

$$\exp\left(\int_0^t \frac{1}{\mathrm{e}^{-s}q-1}ds\right) = \frac{q-1}{\mathrm{e}^{-t}q-1}\frac{\mathrm{e}^{-t}q}{q},$$

which entails that

$$\exp\left(\int_0^t \kappa_R(\mathrm{e}^{-s}q)ds\right) = \frac{\Gamma(q+1)}{\Gamma(\mathrm{e}^{-t}q+1)}\frac{q-1}{\mathrm{e}^{-t}q-1}\frac{\mathrm{e}^{-t}q}{q} = \frac{q-1}{\mathrm{e}^{-t}q-1}\frac{\Gamma(q)}{\Gamma(\mathrm{e}^{-t}q)},$$

so (4.5.1) follows from Theorem 4.2.9.

For the reader's convenience, we give the proof of Proposition 4.5.1.

Proof of Proposition 4.5.1. Let ξ be a SNLP with characteristics $(0, -\gamma+1, \Lambda, 0)$, where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant, and the Lévy measure Λ has density

$$\Lambda(dz) = \mathrm{e}^z(1-\mathrm{e}^z)^{-2}dz, \quad z \in (-\infty, 0).$$

We know from [10] that the Laplace exponent of ξ is $\Phi_R(q) := q\psi(q+1)$ [2]. We also have that

$$\int_{-\infty}^{0} (1-\mathrm{e}^z)^q \mathrm{e}^z (1-\mathrm{e}^z)^{-2} dz = \frac{1}{q-1}, \quad q > 1.$$

So ξ has cumulant κ_R.

Write P_x for an exponential OU type process X with characteristics $(\Phi_R, 1)$ starting from $x > 0$, then we shall prove that $\mathbf{X}^R$ is Markovian growth-fragmentation associated with X. In this direction, let us consider a cell system $\mathcal{X}$ described as follows. Let the Eve process be $\mathcal{X}_\emptyset := W_1$, which has distribution P_1 by Theorem 3.1 in [10]. At each time $\mathcal{X}_\emptyset$ has a jump, Π has a dislocation. Say at time $s > 0$, the block $\Pi_1(s)$ splits into B_1 and B_2, with B_1 being the block that contains 1. Write Π^{B_2} for the fragmentation process constrained to B_2 and let $y := \lim_{n\to\infty} n^{-\mathrm{e}^{-s}} \#\{i \le n \ : \ i \in B_2\}$. Given y and s, we find by using Proposition 2.3 and Theorem 3.1 in [10] that the conditional distribution of the weight process

$$W_1^{B_2}(t) := \lim_{n\to\infty} n^{-\mathrm{e}^{-(t+s)}} \#\{i \le n \ : \ i \in \Pi_1^{B_2}(t+s)\}, \quad t \ge 0$$

has law P_y. We thus view $W_1^{B_2}$ as the child cell process born at time s. In this way we find all the daughter processes of the first generation, these daughter processes being independent one of the others. By iteration of this argument, we obtain a cell system driven by X, which means that $\mathbf{X}^R$ is Markovian growth-fragmentation associated with X. So we know from Proposition 4.4.7 that $\mathbf{X}^R$ is a binary OU type growth-fragmentation process with characteristics $(\kappa_R, 1)$. $\square$

[2] The Lévy-Khintchine formula in [10] has a compensation term different from (4.2.1), so the drift coefficient is changed.

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